Problem 1: (10 points)
\textit{Experiment 5: Measuring Moment of Inertia Pre-Lab Question 1}

A steel washer is mounted on a cylindrical rotor of radius $r = 12.7 \text{ mm}$. A massless string, with an object of mass $m = 0.055 \text{ kg}$ attached to the other end, is wrapped around the side of the rotor and passes over a massless pulley.

Assume that there is a constant frictional torque about the axis of the rotor. The object is released and falls. As the mass falls, the rotor undergoes an angular acceleration of magnitude $\alpha_1$. After the string detaches from the rotor, the rotor coasts to a stop with an angular acceleration of magnitude $\alpha_2$. Let $g = 9.8 \text{ m} \cdot \text{s}^{-2}$ denote the gravitational constant. Based on the data in the figure below, what is the moment of inertia $I_R$ of the rotor assembly (including the washer) about the rotation axis?
Solution:

A key point to keep in mind is that both \( \alpha_1 \) and \( \alpha_2 \) are given as magnitudes, in both the symbolic and numerical calculations. Along the same lines, use \( \tau_f \) as the magnitude of the frictional torque.

While the mass is falling, the rotor/washer combination has a net torque due to the tension in the string and the friction torque, and setting the net torque equal to the rate of change of angular momentum,

\[
Tr - \tau_f = I_R \alpha_1
\]  

(1.1)

The tension is related to the weight \( mg \) of the falling object by

\[
mg - T = ma_1 = ma_1 r
\]  

(1.2)

where \( a_1 = \alpha_1 r \) has been used to express the linear acceleration of the object to the angular acceleration of the rotor; that is, the string does not stretch.

Before proceeding, it might be illustrative to multiply Equation (1.2) by \( r \) and add to Equation (1.1) to obtain

\[
mg - \tau_f = (I_R + mr^2) \alpha_1
\]  

(1.3)

Equation (1.3) contains the unknown frictional torque, and this torque is determined by considering the slowing of the rotor/washer after the string has detached. The net torque on the system is just this frictional torque, and so
\[-\tau_f = -I_R\alpha_2 \quad (1.4)\]

Note that in Equation (1.4), \(\tau_f\) and \(\alpha_2\) have the same sign, consistent with the assumption that both are positive.

Subtracting Equation (1.4) from Equation (1.3) eliminates \(\tau_f\), and solving for \(I_R\) yields

\[I_R = \frac{mr(g - r\alpha_1)}{\alpha_1 + \alpha_2} \quad (1.5)\]

For a numerical result, the values for \(\alpha_1\) and \(\alpha_2\) from the above figure need to be found. An eyeball look (you’ll do better than this when you do the lab – that’s what LabView is for) gives \(\alpha_1 = \left(96 \text{ rad} \cdot \text{s}^{-1}\right)/(1.15 \text{ s})\) and \(\alpha_2 = \left(89 \text{ rad} \cdot \text{s}^{-1}\right)/(2.85 \text{ s})\). Inserting these expressions (the intermediate calculation was not done in the program that generated the following result),

\[I_R = 5.3 \times 10^{-2} \text{ kg} \cdot \text{m}^2. \quad (1.6)\]

Problem 2 (10 points):

*Experiment 5: Measuring Moment of Inertia Pre-Lab Question*

In the angular momentum experiment, shown to the right, a washer is dropped smooth side down onto the spinning rotor.

The graph below shows the rotor angular velocity \(\omega\left(\text{rad} \cdot \text{s}^{-1}\right)\) as a function of time.

Assume the following:
- The rotor and washer have the same moment of inertia \(I\).
- The friction torque \(\tilde{\tau}_f\) on the rotor is constant during the measurement.
a) Find an expression for the magnitude $|\vec{\tau}_f|$ in terms of $I$ and numbers you obtain from the graph.

b) What torque does the rotor exert on the washer during the collision (between $t = 1.90 \, \text{s}$ and $t = 2.40 \, \text{s}$)? Express your answer in terms of $I$ and numbers you obtain from the graph.

c) How many radians does the rotor rotate during the collision (between $t = 1.90 \, \text{s}$ and $t = 2.40 \, \text{s}$)? Give a numerical answer.

d) How many radians does the washer rotate during the collision (between $t = 1.90 \, \text{s}$ and $t = 2.40 \, \text{s}$)? Give a numerical answer.

Note: express all of your answers in terms of $I$ and numbers you obtain from the graph. Be sure to give an analytic expression prior to substituting the numbers from the graph.

Solutions:

a) First, make sure that the problem makes sense. Between times $t = 0.40 \, \text{s}$ and $t = 1.90 \, \text{s}$, the magnitude of the angular acceleration is $\Delta \omega / \Delta t = 40 \, \text{rad} \cdot \text{s}^{-2}$ and between times $t = 2.40 \, \text{s}$ and $t = 4.40 \, \text{s}$ the magnitude of the angular acceleration is $\Delta \omega / \Delta t = 20 \, \text{rad} \cdot \text{s}^{-2}$. During these two time intervals, the only torque is the friction torque, assumed constant, and doubling the net moment of inertia halves the angular acceleration.
We then have $|\vec{\tau}_f| = I(40 \, \text{rad} \cdot \text{s}^{-2})$. This is also $|\vec{\tau}_f| = 2I(20 \, \text{rad} \cdot \text{s}^{-2})$, but that’s not part of this problem, just a consistency check.

b) The main point to recognize in this part of the problem is that the washer accelerates from an initial angular velocity of zero to the final angular velocity shown on the graph.
The only torque on the washer is the torque that the rotor exerts on the washer, and the magnitude of this torque is

\[
\tau_{\text{rotor-washer}} = I \frac{\Delta \omega_{\text{washer}}}{\Delta t} = I \frac{100 \text{ rad} \cdot \text{s}^{-1}}{0.50 \text{ s}} = I \left(200 \text{ rad} \cdot \text{s}^{-2}\right)
\]  

(2.1)

c) Let’s go the simple way, and say that the angle through which the rotor rotates is the product of the average angular velocity and the time interval,

\[
\Delta \theta = \omega_{\text{ave}} \Delta t = (160 \text{ rad} \cdot \text{s}^{-1})(0.50 \text{ s}) = 80 \text{ rad} ,
\]  

(2.2)

about 50 revolutions.

d) We’ve done part c) simply, so let’s do the same for the washer. Here, the average angular velocity is 50 rad \cdot s^{-1} over the same time interval, or 25 rad.

**Problem 3 (10 points): Conservation of Angular Momentum**

A meteor of mass \( m = 2.1 \times 10^{13} \text{ kg} \) is approaching earth as shown on the sketch. The distance \( h \) on the sketch below is called the impact parameter. The radius of the earth is \( R_e = 6.37 \times 10^6 \text{ m} \). The mass of the earth is \( m_e = 5.98 \times 10^{24} \text{ kg} \). Suppose the meteor has an initial speed of \( v_0 = 1.0 \times 10^7 \text{ m} \cdot \text{s}^{-1} \). Assume that the meteor started very far away from the earth. Suppose the meteor just grazes the earth. You may ignore all other gravitational forces except the earth. Find the impact parameter \( h \).

![Sketch of meteor approaching Earth](image)

As will be seen this problem is best done symbolically, with numerical values used at the end of the calculations. We’ll also need to neglect any air resistance when the meteor approaches the earth.

As the problem statement implies, we will need to conserve angular momentum. The meteor’s mass is so much small than the mass of the earth that we will assume that the earth’s motion is not affected by the meteor, and so the angular momentum about the
center of the earth is a constant. Using \( v_a \) for the speed of the meteor at its nearest approach to the earth, we have

\[
m v_0 h = mv_a R_e
\]  
(3.1)

In order to solve for \( h \), we need to find \( v_a \). Since we are neglecting all forces on the meteor other than the earth’s gravity, mechanical energy is conserved, and

\[
\frac{1}{2} mv_0^2 = \frac{1}{2} mv_a^2 - \frac{G mm_e}{R_e}
\]  
(3.2)

where we have taken the meteor to have speed \( v_0 \) at a distance “very far away from the earth” to mean that we neglect any gravitational potential energy in the meteor-earth system.

The parameter \( m \) is canceled from both Equations (3.1) and (3.2). Solving Equation (3.1) for \( v_a \) and substitution into Equation (3.2) gives

\[
v_0^2 = v_0^2 \left(\frac{h}{R_e}\right)^2 - \frac{2Gm_e}{R_e}
\]  
(3.3)

and solving for the parameter \( h \),

\[
h = \sqrt{R_e^2 + \frac{2Gm_e R_e}{v_0^2}}
\]  
(3.4)

and inserting numerical values, with \( G = 6.67 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} \) yields \( h = 7.1 \times 10^9 \text{ m} \), roughly ten times the earth’s radius.

**Problem 4 (10 points)**

**V-Groove Frictional Torque and Fixed Axis Rotation**

A cylinder of mass \( m \) and radius \( R \) is rotated in a V-groove with constant angular velocity \( \omega_0 \). The coefficient of friction between the cylinder and the surface is \( \mu \). What external torque must be applied to the cylinder to keep it rolling?
Solution:

A choice of coordinate system is essential. There are many ways to do this problem, and several ways to select a coordinate system. The solution presented here refers to vertical and horizontal forces. Any way the problem is done, it must be recognized that there are two contact forces, \( \mathbf{C}_1 \) at the left and \( \mathbf{C}_2 \) at the right. Each contact force has two constituents, a normal force (\( \mathbf{N}_1 \) or \( \mathbf{N}_2 \)) with upward vertical components and a frictional force (\( \mathbf{f}_1 \) or \( \mathbf{f}_2 \)), each with a horizontal component directed to the left in the figure. (In an attempt to keep the figure to scale, the vector representing \( \mathbf{f}_2 \) in the figure below is more or less just an arrowhead.)

![Diagram of forces](image)

Given the coefficient of friction, we have \( f_1 = \mu N_1 \), \( f_2 = \mu N_2 \).

The horizontal force components must add to zero;

\[
N_1 \frac{1}{\sqrt{2}} - f_1 \frac{1}{\sqrt{2}} - N_2 \frac{1}{\sqrt{2}} - f_2 \frac{1}{\sqrt{2}} = 0
\]

\[
N_1 \frac{1}{\sqrt{2}} (1 - \mu) - N_2 (1 + \mu) \frac{1}{\sqrt{2}} = 0.
\]

The vertical components of the normal forces must sum to the weight of the cylinder;
The second expressions Equations (4.1) and (4.2) are two linear equations in the unknowns $N_1$ and $N_2$. There are many ways to solve such pairs of equations; the first method presented here is perhaps pedestrian, but it works. Multiply (4.1) by $1 - \mu$ and (4.2) by $1 + \mu$ and add to eliminate the terms containing $N_2$; the result is

\[ \frac{N_1}{\sqrt{2}} \left((1 - \mu)^2 + (1 + \mu)^2\right) = mg(1 + \mu) \]

(4.3)

Similarly, multiplying (4.1) by $1 + \mu$ and (4.2) by $1 - \mu$ and subtracting to eliminate the terms containing $N_1$ gives

\[ N_2 = \frac{mg}{\sqrt{2}} \frac{1 - \mu}{1 + \mu^2}. \]

(4.4)

The magnitude of the frictional torque is then seen to be

\[ \tau_{\text{fric}} = (f_1 + f_2)R = \mu(N_1 + N_2)R = \mu\left(\frac{mg}{\sqrt{2}} \frac{2}{1 + \mu^2}\right)R = mgR\sqrt{2} \frac{\mu}{1 + \mu^2}. \]

(4.5)

and this must also be the magnitude of the applied torque to maintain constant angular velocity.

Note that the moment of inertia of the cylinder, that is, whether the cylinder is hollow, solid, or something in between, does not enter the solution of this problem.

** Alternate Solution: **

Not really an alternate solution, but different algebra.

Re-express the second expressions in each of (4.1) and (4.2) as
\[ (N_1 - N_2) - \mu(N_1 + N_2) = 0 \]
\[ (N_1 + N_2) + \mu(N_1 - N_2) = \sqrt{2} mg. \]  

Multiply the first equation in (4.6) by \(-\mu\) and add to the second to obtain
\[ (N_1 + N_2)(1 + \mu^2) = \sqrt{2} mg, \]
which leads to the results in (4.5).

Similarly, in the first expressions in Equations (4.1) and (4.2) we could have substituted \(N_1 = f_1 / \mu, \ N_2 = f_2 / \mu\) and solved for the two friction forces directly.

**Problem 5: (10 points)**

*Wheel, inclined plane, two masses and a rope*

A wheel in the shape of a uniform disk of radius \(R\) and mass \(m_p\) is mounted on a frictionless horizontal axis. The wheel has moment of inertia about the center of mass \(I_{cm} = (1/2)m_pR^2\). A massless cord is wrapped around the wheel and one end of the cord is attached to an object of mass \(m_1\) that can slide up or down a frictionless inclined plane. The other end of the cord is attached to a second object of mass \(m_2\) that hangs over the edge of the inclined plane. The plane is inclined from the horizontal by an angle \(\theta\). Once the objects are released from rest, the cord moves without slipping around the disk. Find the accelerations of each object, and the tensions in the string on either side of the pulley.

**Solution:**

For this problem, the wheel is not massless, and so as the blocks move, the wheel moves, and as the blocks accelerate the wheel has an angular acceleration and hence a net torque. The wheel axle is given as being frictionless, so the torque on the wheel is due to the difference between the tensions in the two parts of the rope. Denote these tensions as \(T_1\) and \(T_2\).
The equation of motion for the suspended object is simple,

\[ m_1 g - T_1 = m_1 a. \]  \hspace{1cm} (5.1)

The equation of motion for the object on the incline involves choosing a coordinate system with one coordinate directed up the incline. The net force in this direction is the tension \( T_2 \) and the component of gravity \( -m_2 g \sin \theta \), leading to

\[ T_2 - m_2 g \sin \theta = m_2 a. \]  \hspace{1cm} (5.2)

The relation between the torque on the wheel and its angular acceleration \( \alpha \) is

\[ (T_1 - T_2) R = I \alpha. \]  \hspace{1cm} (5.3)

At this point, note that Equation (5.1) assumes a positive acceleration downward, Equation (5.2) assumes a positive direction up the incline and Equation (5.3) assumes a positive angular acceleration in the counterclockwise direction in the figure. These direction choices are consistent.

Equations (5.1), (5.2) and (5.3) are five linear equations in five unknowns, and of course we need to impose further conditions. The simplest is \( a_i = a_s = a \) with the chosen sign convention, indicating that the cord doesn’t stretch. If the cord doesn’t slip on the wheel, the angular acceleration is \( \alpha = a / R \). Equations (5.1), (5.2) and (5.3) then become

\[ m_1 g - T_1 = m_1 a \]
\[ T_2 - m_2 g \sin \theta = m_2 a \]
\[ T_1 - T_2 = 4a / R^2. \]  \hspace{1cm} (5.4)

In this form, the two tensions may be eliminated by adding the three expressions in (5.4), with the result

\[ m_1 g - m_2 g \sin \theta = a (m_1 + m_2 + I / R^2) = a (m_1 + m_2 + m_p / 2) \]
\[ a = g - m_1 - m_2 \sin \theta \]
\[ m_1 + m_2 + m_p / 2. \]  \hspace{1cm} (5.5)

Note that this result does not depend on the radius \( R \). Also note that if \( m_2 \sin \theta > m_1 \), \( a < 0 \), and the object on the incline will accelerated down the incline, and the suspended block will accelerate upwards. For any combination of the parameters, \( |a| < g \).

The tensions are found by substituting the acceleration found in (5.5) into the respective equations (5.4). There is some algebra involved, but not much;
\[ T_i = m_i (g - a) \]
\[ = m_i g \left[ 1 - \frac{m_1 - m_2 \sin \theta}{m_1 + m_2 + m_p/2} \right] \quad (5.6) \]
\[ = m_i g \frac{m_2 (1 + \sin \theta) + m_p/2}{m_1 + m_2 + m_p/2}, \]

\[ T_2 = m_2 a + m_2 g \sin \theta \]
\[ = m_2 g \left[ \frac{m_1 - m_2 \sin \theta}{m_1 + m_2 + m_p/2} + \sin \theta \right] \quad (5.7) \]
\[ = m_2 g \frac{m_1 (1 + \sin \theta) + m_p \sin \theta/2}{m_1 + m_2 + m_p/2}. \]

Any further simplification might not be useful. However, note that if \( m_2 \sin \theta > m_1 \), \( T_i > m_2 g \), consistent with the suspended block accelerating upward.

Of course, the relations in (5.4) could be manipulated algebraically to eliminate one tension and the acceleration \( a \) in terms of the other tension, but that alternate solution won’t be considered here.