Problem 1 (10 points): A hollow cylinder of outer radius \( R \) and mass \( M \) with moment of inertia about the center of mass \( I_{cm} = MR^2 \) starts from rest and moves down an incline tilted at an angle \( \theta \) from the horizontal. The center of mass of the cylinder has dropped a vertical distance \( h \) when it reaches the bottom of the incline. Let \( g \) denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is \( \mu_s \). The cylinder rolls without slipping down the incline. The goal of this problem is to find an expression for the smallest possible value of \( \mu_s \) such that the cylinder rolls without slipping down the incline plane and the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.

a) Write down a plan for solving this problem. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any diagrams or sketches that you plan to use.

b) What is the minimum value for the coefficient of static friction \( \mu_s \) such that the cylinder rolls without slipping down the incline plane? Express your answer in terms of \( M \), \( R \), \( g \), \( \theta \) and \( h \) as needed.

c) What is the magnitude of the velocity of the center of mass of the cylinder when it reaches the bottom of the incline? Express your answer in terms of \( M \), \( R \), \( g \), \( \theta \) and \( h \) as needed.
Solutions:

a) The two parts of the problem are seen to be distinct; find the minimum value of $\mu_s$, and from the resulting acceleration find the speed of the cylinder when it reaches the bottom of the incline. To find the minimum value of $\mu_s$, we will need to know something about the forces and torques, specifically the relation between the friction force on the normal force; that is, the components of the contact force. As a result of this determination, we will find the acceleration and hence the speed at the bottom of the incline. A figure showing the forces is shown below. (The figure is from an in-class presentation from last year, but we’re sure not going to waste a good figure.)

![Diagram of forces](image)

b) With the coordinates system shown, Newton’s Second Law, applied in the $x$- and $y$-directions in turn, yields

\[
\begin{align*}
Mg \sin \theta - f &= Ma \\
N - Mg \cos \theta &= 0.
\end{align*}
\]

The equations in (1.1) represent two equations in three unknowns, and so we need one more relation. As described in part (a), this will come from torque considerations.

Of course, any point could be used for the origin in computing torques, but the “obvious” choice of the center of the cylinder turns out to make things easiest (judgment call, of course). Then, the only force exerting a torque is the friction force, and so we have
\[ f R = I_{cm} \alpha = M R^2 \left( \frac{a}{R} \right) = M a, \quad (1.2) \]

where \( I_{cm} = M R^2 \) and the kinematic constraint for the no-slipping condition \( \alpha = a/R \) have been used. Equation (1.2) leads to \( f = M a \), and inserting this into the first expression in (1.1) gives the two relations

\[
\begin{align*}
  f &= \frac{1}{2} M g \sin \theta \\
  a &= \frac{1}{2} g \sin \theta.
\end{align*}
\]

(1.3)

We're still not done. For rolling without slipping, we need \( f < \mu_s N \), so we need, using the second expression in (1.1),

\[ \mu_s > \frac{1}{2} \tan \theta. \]

(1.4)

c) The cylinder rolls a distance \( L = h/\sin \theta \) down the incline, and the speed \( v_t \) at the bottom is related to the acceleration found in part (c) by

\[
\begin{align*}
  v_t^2 &= 2aL = 2 \left( \frac{1}{2} g \sin \theta \right) \left( \frac{h}{\sin \theta} \right) \\
  &= gh.
\end{align*}
\]

(1.5)

The result of Equation (1.5) can and should be checked by energy conservation (for rolling without slipping, the friction force does no mechanical work). For the given moment of inertia, the final kinetic energy is

\[
K_f = \frac{1}{2} M v_t^2 + \frac{1}{2} I_{cm} \omega_t^2 \\
= \frac{1}{2} M v_t^2 + \frac{1}{2} M R^2 \left( \frac{v_t}{R} \right)^2 \\
= M v_t^2,
\]

(1.6)

and setting the final kinetic energy equal to the loss of gravitational potential energy leads to Equation (1.5).
Problem 2 (10 points): A bowling ball of mass $m$ and radius $R$ is initially thrown down an alley with an initial velocity $v_0$ and backspin with magnitude $\omega_0$, such that $v_0 > R\omega_0$. The moment of inertia of the ball about its center of mass is $I_{\text{cm}} = \frac{2}{5}mR^2$. You are going to try to find the velocity $v_f$ of the bowling ball when it just starts to roll without slipping.

a) Write up your plan for solving this problem. You may find some of the following concepts useful: angular impulse is equal to the change in angular momentum; linear impulse is equal to the change in momentum; Newton’s Second Law; torque about the center-of-mass is proportional to the angular acceleration; if the torque about a point is zero, then the angular momentum about that point is constant; etc.

b) What is the velocity $v_f$ of the bowling ball when it just starts to roll without slipping?

Solutions:

a) Clearly, angular momentum will be needed to be considered to solve this problem. Note that we are not given anything about the nature of the force, and experienced bowlers know that modeling the surface as having a constant coefficient of friction is not realistic. (Candlepins don’t really count.) So, if we take the origin for determining torques and angular momenta about a point where the friction exerts no torque, we shouldn’t need to know about the nature of the friction force. Accordingly, choose the origin to be the original point of contact of the ball with the lane surface. Subsequently, even though the ball has moved, friction will still exert no torque.

b) With respect to the contact point on the ground, the initial and final angular momenta are both the sum of two terms, one representing the motion of the center of mass and the other the rotation (“spin”) of the ball;

\[
L_{\text{initial}} = mv_0R - I_{\text{cm}}\omega_0 \\
L_{\text{final}} = mv_fR + I_{\text{cm}}\omega_f.
\]  

The problem is now one of basic algebra. For rolling without slipping, $\omega_f = v_f / R$, and the given $I_{\text{cm}} = \frac{2}{5}mR^2$ gives
\[ v_t = \left(5v_0 - 2\omega_0 R\right)/7. \]  

(2.2)

It’s important to note the signs in the expressions in (2.1). We are given (and the figure certainly implies) that the scalar quantity \( \omega_0 \), representing backspin, is positive, and so with positive direction for angular momenta being clockwise, the \( \omega_0 \) term in the initial angular momentum is negative.

This problem may of course be done by considering torques and angular momenta about the center of the ball. The change in linear momentum (watch the signs again) is the impulse

\[ \Delta p = m(v_t - v_0) = -\int f \, dt \]  

(2.3)

and the change in angular momentum is the angular impulse (the signs are still important)

\[ \Delta L = I_{cm}(\omega_f + \omega_0) = \int Rf \, dt. \]  

(2.4)

Eliminating the linear impulse \(-\int f \, dt\) between Equations (2.3) and (2.4), and using the given \( I_{cm} = (2/5)mR^2 \) yields the same result as that in Equation (2.2).
Problem 3 (10 points): Billiards Challenge

A spherical billiard ball of uniform density has mass $m$ and radius $R$ and moment of inertia about the center of mass $I_{cm} = (2/5)mR^2$. The ball, initially at rest on a table, is given a sharp horizontal impulse by a cue stick that is held an unknown distance $h$ above the centerline (see diagram below). The coefficient of sliding friction between the ball and the table is $\mu_k$. You may ignore the friction during the impulse. The ball leaves the cue with a given speed $v_0$ and an unknown angular velocity $\omega_0$. Because of its initial rotation, the ball eventually acquires a maximum speed of $(9/7)v_0$. The point of the problem is to find the ratio $h / R$.

![Billiards Challenge Diagram](image)

a) Write up your plan for solving this problem. You may find some of the following concepts useful: angular impulse is equal to the change in angular momentum; linear impulse is equal to the change in momentum; Newton’s Second Law; torque about the center-of-mass is proportional to the angular acceleration; if the torque about a point is zero, then the angular momentum about that point is constant; etc.

b) Find the ratio $h / R$.

Solutions:

a) There are several ways to approach this problem. The method presented here avoids any calculation of the force or torque provided by friction, or the details of the force between the cue and the ball. This method will first consider the “collision” between the cue and the ball by taking the collision point as the origin for finding the angular momentum, as the force between the cue and the ball exerts no torque about this point, and we are given that the friction may be ignored during this interaction. After this collision, the angular momentum will be taken about the initial contact point between the ball and the felt. (It should be noted that this method anticipates the answer, which does not involve the coefficient of friction $\mu_k$ and also relies on having done Problem 2 above.) It will be helpful to infer, either from the figure and from the fact that $v_f > v_0$, that the ball is given overspin.
b) With respect to the point where the cue is in contact with the ball, note that the rotational angular momentum and the angular momentum due to the motion of the center of mass have different signs; the former is clockwise and the latter is counterclockwise. The sum of these contributions to the angular momenta must sum to zero, and hence have the same magnitude;

\[ I_{\text{cm}} \omega_0 = m v_0 \ h. \]  

(3.1)

While the ball is rolling and slipping, angular momentum is conserved about the contact between the ball and the felt. The initial and final angular momenta are

\[ L_{\text{initial}} = m v_0 R + I_{\text{cm}} \omega_0 \]
\[ = m v_0 (R + h) \]
\[ L_{\text{final}} = m v_f R + I_{\text{cm}} \omega_f \]
\[ = m v_f R + (2/5)(mR^2)(v_f / R) \]
\[ = (7/5)m v_f R \]
\[ = (9/5)m v_0 R, \]  

(3.2)

where Equation (3.1) and the given relations \( I_{\text{cm}} = (2/5)mR^2 \) and \((9/7)v_0 \) have been used. Setting the initial and final angular momenta equal and solving for \( h / R \) gives

\[ \frac{h}{R} = \frac{4}{5} \]  

(3.3)

(note that the figure is not quite to scale).

As an alternative, taking the angular momentum after the collision about the center of the ball, note that the time \( \Delta t \) between the moments the ball is struck and when it begins to roll without slipping is \( \Delta v / (\mu_c g) \). But, if the angular momentum is taken about the center of the ball, after the ball is struck the angular impulse delivered to the ball by the friction force is

\[ (\mu_c mg) R \Delta t = I_{\text{cm}} \ (\omega_f - \omega_0). \]  

(3.4)

Eliminating \( \Delta t \) between these expressions leads to the same result obtained by equating the first and third expressions in (3.2).
Problem 4 (10 points):

Three point-like objects located at the points A, B and C of respective masses \( M_A = M \), \( M_B = M \) and \( M_C = 2M \), as shown in the figure below. The three objects are initially oriented along the \( y \)-axis and connected by rods of negligible mass each of length \( D \), forming a rigid body. A fourth object of mass \( M \) moving with velocity \( \mathbf{v}_0 \) \( \hat{i} \) collides and sticks to the object at rest at point A. Neglect gravity. Give all your answers in terms of \( M \), \( v_0 \) and \( D \) as needed. The \( z \)-axis points out of the page.

![Diagram of the system](image)

a) Describe qualitatively in words how the system moves after the collision: direction, translation and rotation.

b) What is the direction and magnitude of the linear velocity of the center of mass after the collision?

c) What is the magnitude of the angular velocity of the system after the collision?

d) What are the directions and magnitudes of the velocity \( \mathbf{v}_c \) and acceleration \( \mathbf{a}_c \) of the object located at the point C immediately after the collision?

Solutions:

a) From conservation of linear momentum, the system will move to the right (in the positive \( x \)-direction). The system will rotate about its center of mass, clockwise (in the negative \( z \)-direction from the right-hand rule).

b) The position of the center of mass of the system is at the initial position of object B. The velocity of the center of mass is
\[ \ddot{v}_{\text{cm}} = \frac{Mv_0 \hat{i}}{M + (M + M + 2M)} = \frac{1}{5} v_0 \hat{i} \]  

(4.1)

and this will be the velocity of the center of mass after the collision.

c) We could of course choose any point about which to calculate the center of mass. Since the system after the collision is symmetric about the object at point B, choosing this point as the origin will simplify calculations. (The choice of this point as the origin is strongly suggested by the diagram.) The initial and final angular momenta about this point are

\[ \hat{L}_{\text{initial, B}} = Mv_0 D (-\hat{k}) \]
\[ \hat{L}_{\text{final, B}} = I_{\text{cm, B}} \omega_f (-\hat{k}) \]  

(4.2)

where \( I_{\text{cm, B}} = 2(2MD^2) \) is the moment of inertia of the system about point B. Equating initial and final angular momenta yields \( \omega_f = v_0 / (4D) \).

d) The velocity is found by adding the center of mass velocity of the system to the velocity of the mass at point C relative to the center of mass. The velocity of the center of mass is that found in part (b) and the velocity with respect to the center of mass is given by the cross product of the vector angular velocity and the vector displacement of point C from the center, or

\[ \ddot{v}_C = \left( \frac{1}{5} v_0 \hat{i} \right) + \left( -\frac{v_0}{4D} \hat{k} \right) \times (-D\hat{j}) = -\frac{1}{20} v_0 \hat{i}. \]  

(4.3)

In the limit of the collision being instantaneous, immediately after the collision the rod attaching the object at point C to the center of mass is parallel to the \( \hat{j} \)-direction. Viewed from the center of mass, this must be the direction of the acceleration. To find the magnitude of the acceleration, we also go to the center of mass frame and use for the speed of the object \( v_{\text{C, cm}} = v_C + v_{\text{cm}} = v_0 / 4 \) (the magnitudes are added because the vector velocities are oppositely directed) and so the vector acceleration is

\[ \ddot{a}_C = \frac{v_{\text{C, cm}}^2}{D} \hat{j} = \frac{v_0^2}{16D} \hat{j}. \]  

(4.4)

This acceleration will be the same in both the “lab” frame and the center of mass frame, even though the speed of the object is different in the different frames.
Problem 5 (10 points): A physical pendulum consists of a disc of radius \( R \) and mass \( m_d \) (shown as “\( M \)” in the figure below) fixed at the end of a rod of mass \( m_r \) and length \( l \).

\[ \text{Solutions:} \]

a) Find the period of the pendulum.

b) How does the period change if the disk is mounted to the rod by a frictionless bearing so that it is perfectly free to spin?

\[ I_{\text{cm}} = \frac{m_r (l/2) + m_d l}{m_r + m_d} = \frac{m_r l}{2} + m_d \]

from the pivot and moment of inertia

\[ I_{\text{pivot}} = m_r \frac{l^2}{3} + m_d \left( l^2 + \frac{R^2}{2} \right) \]
Any further simplification is of limited utility.

b) If the disc is free to rotate, the system cannot be treated as a simple pendulum. Qualitatively, using energy considerations, the rotational kinetic energy of the disc about its center does not change as the pendulum moves, and so this constant term would be treated as such in any conservation of energy relations. An effect of this is that the pendulum is swinging faster when passing through the vertical than it would be for the fixed disc, consistent with a smaller period.

Or, for the case when the disc is fixed there must be some torque exerted on the disc by the rod by whatever is holding the disc fixed, and hence a torque exerted on the rod by the disc. When this internal torque is removed, the pendulum rod is “less restricted” and hence should oscillate with a smaller period.

Mathematically, by not including the rotation of the disc about its center, we set $R = 0$, with the result

$$T = 2\pi \sqrt{\frac{l}{g}} \sqrt{\frac{m_r/3 + m_d}{m_r/2 + m_d}}.$$  

(4.8)

Note that the mass of the disc still appears; the disc bearings will still exert a torque on the rod, and the center of the disc still moves with the rod, but terms reflecting rotation are not present.