Problem 1: (10 Points)

Part a) (2 points) A gyroscope has a wheel at one end of an axle, which is pivoted at point \( O \) as shown in the figure. The wheel spins about the axle in the direction shown by the arrow in the figure. At the moment shown in the figure, the axle is horizontal and in the plane of the page. Let \( \mathbf{L} \) be the angular momentum of the gyroscope about the center of mass of the gyroscope. You may ignore the mass of the axle and assume the spin angular velocity is much greater than the precessional angular velocity.

The direction of the vector \( d\mathbf{L}/dt \) of the gyroscope at the moment shown in the figure is:

1. \(+\hat{i}\) direction.
2. \(-\hat{j}\) direction.
3. \(-\hat{j}\) direction.
4. \(+\hat{k}\) direction.
5. \(-\hat{k}\) direction.

Explain your reasoning.

Answer: 5, the \(-\hat{k}\) direction (or, \(-\hat{k}\) in the notation of the figure).

By the right-hand rule, the angular momentum is radially inward, the \(-\hat{i}\) direction in the figure. Taking the torque about \( O \), the net torque is due to the weight of the gyroscope, and if \( \mathbf{R} \) is the position vector from point \( O \) to the center of the wheel, \( \mathbf{R} \times m\mathbf{g} \) is into the page of the figure, the \(-\hat{k}\) direction. Although not part of this problem, the gyroscope will precess in such a way as to move out of the page of the figure.
A valid alternative is to take torques on the wheel/axle combination about the center of mass of the gyroscope. Then, the only force exerting a torque is the support force at point $O$, and this force is upward and equal in magnitude to the weight. The position vector from the center of mass to $O$ is $-\mathbf{R}$, and $\left(-\mathbf{R}\right) \times (-mg) = \mathbf{R} \times mg$; the torque is the same.

**Part b) (8 points)**

A wheel is at one end of an axle of length $l$. The axle is pivoted at an angle $\phi$ with respect to the horizontal. The wheel is set into motion so that it executes uniform precession; that is, the wheel’s center of mass moves with uniform circular motion. The wheel has mass $m$ and moment of inertia $I_{cm}$ about its center of mass. Its spin angular velocity has magnitude $\omega_{\text{spin}}$ and is directed as shown in the figure below. Neglect the mass of the axle. What is the angular frequency that the gyroscope precesses about the vertical axis? Does the gyroscope rotate clockwise or counterclockwise about the vertical axis (as seen from above)?

**Solution:**

Taking torques about the support point, the only force supplying the net torque is the weight of the wheel. The moment arm is $R_\perp = l \cos \phi$ and so the net torque has magnitude $\tau_{\text{net}} = mgl \cos \phi$. 
The direction of this torque, from the right-hand rule, is into the page in the above figure; in vector form this direction would be in the positive $\theta$-direction in cylindrical coordinates, or the $-\hat{k}$ direction in the coordinate system used in part (a) above. Thus, the vertical component of the angular moment does not change. The magnitude of the spin angular momentum is $L_{\text{cm}}^{\text{spin}} = I_{\text{cm}} \omega_{\text{spin}}$ and the horizontal component is $L_{\text{cm}}^{\text{spin}} \cos \phi = I_{\text{cm}} \omega_{\text{spin}} \cos \phi$. The precession frequency $\Omega$ is found from

$$\Omega I_{\text{cm}}^{\text{spin}} \cos \phi = \tau_{\text{net}}$$
$$\Omega I_{\text{cm}} \omega_{\text{spin}} \cos \phi = mgl \cos \phi$$

$$\Omega = \frac{mgl}{I_{\text{cm}} \omega_{\text{spin}}},$$

independent of the angle $\phi$.

In the figure, the direction of the horizontal component of the angular momentum is radially outward, and a torque into the page means that the gyroscope precesses into the page, which would be counterclockwise as seen from above.

**Problem 2 (10 points)**

The effective potential corresponding to a pair of particles interacting through a central force is given by the expression

$$U_{\text{eff}}(r) = \frac{L^2}{2 \mu r^2} + Cr^3$$

where $L$ is the angular momentum, $\mu$ is the reduced mass and $C$ is a constant. The total energy of the system is $E$. The relationship between $U_{\text{eff}}(r)$ and $E$ is shown in the figure, along with an indication of the associated maximum and minimum values of $r$. 
and the minimum allowed energy $E_{\text{min}}$. In what follows, assume that the center of mass of the two particles is at rest.

a) Find an expression for the radial component $f(r)$ of the force between the two particles. Is the force attractive or repulsive?

b) What is the radius $r_0$ of the circular orbit allowed in this potential? Express your answer as some combination of $L$, $C$, and $\mu$.

c) When $E$ has a value larger than $E_{\text{min}}$, find how rapidly the separation between the particles is changing, $\frac{dr}{dt}$, as the system passes through the point in the orbit where $r = r_0$. Give your answer in terms of some combination of $E$, $E_{\text{min}}$, $L$, $C$, $\mu$ and $r_0$.

d) Does the relative motion between the particles stop when $r = r_{\text{max}}$? If not, what is the total kinetic energy at that point in terms of some combination of $E$, $L$, $C$, $\mu$, $r_{\text{max}}$ and $r_{\text{min}}$?

Solutions:

a) The effective potential is given by

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r),$$  \hspace{1cm} (2.2)

and $f(r) = -\frac{dU}{dr}$ for a central force, and so

$$f(r) = -\frac{d}{dr} U(r) = -\frac{d}{dr} Cr^3 = -3Cr^2.$$  \hspace{1cm} (2.3)

From the figure, $C > 0$, so $f(r) < 0$, a restoring force.

b) The circular orbit will correspond to the minimum effective potential; at this radius the kinetic energy will have no contribution from any radial motion. This minimum effective potential, and hence the radius of the circular orbit, is found from basic calculus and algebra,
\[
\left[ \frac{d}{dr} U_{\text{eff}}(r) \right]_r = r_0 = - \frac{L^2}{\mu r_0^3} + 3Cr_0^2
\]
\[r_0 = \frac{L^2}{3\mu C}, \quad r_0 = \left( \frac{L^2}{3\mu C} \right)^{1/5}.
\]

(c) Recall that the kinetic energy is
\[K = \frac{L^2}{2\mu r^2} + \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2.
\]
The difference \(E - E_{\text{min}}\) is then found by evaluating \(U_{\text{eff}}\) at \(r = r_0\),
\[E - E_{\text{min}} = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2_{r=r_0},
\]
or \(|dr/dt| = \sqrt{(2/\mu)(E - E_{\text{min}})}\).

(d) No; \(dr/dt = 0\), but the kinetic energy, from Equation (2.5), is
\[K_{\text{min}} = \frac{L^2}{2\mu r_{\text{max}}^2}.
\]

**Problem 3 (10 points)**

A pair of particles are interacting through an *attractive* central force whose magnitude is given by the expression
\[f(r) = Cr^4\]
where \(C\) is a positive constant and \(r\) is the relative separation between the particles. The magnitude of the angular momentum associated with the motion is given by \(L\). The reduced mass of the two particles is given by \(\mu\).

(a) What is the potential energy \(U(r)\) associated with this attractive force? Clearly indicate where you have chosen your zero reference potential.

(b) What is the effective potential energy \(U_{\text{eff}}(r)\) associated with this motion? Express your answer as some combination of \(L\), \(C\), \(r\), and \(\mu\). Make a sketch of
the effective potential energy $U_{\text{eff}}(r)$ as a function of $r$, the relative separation between the particles.

c) What is the radius $r_0$ of the circular orbit allowed in this potential?

d) What is the energy $E_{\text{min}}$ associated with this circular orbit? Express your answer as some combination of $L$, $C$, $r_0$, and $\mu$ as needed.

e) When $E$ has a value larger than $E_{\text{min}}$, find how rapidly the separation between the particles is changing, $dr/dt$, as the system passes through the point in the orbit where $r = r_0$. Give your answer in terms of some combination of $E$, $E_{\text{min}}$, $L$, $C$, $\mu$ and $r_0$.

f) Does the relative motion between the particles stop when $r = r_{\text{max}}$? If not, what is the total kinetic energy at that point in terms of some combination of $E$, $L$, $C$, $\mu$, $r_{\text{max}}$ and $r_{\text{min}}$?

Solutions:

Note that for this problem, the constant $C$ is not the same as that in Problem 2.

a) An attractive force means that the radial component of the force must be negative (directed toward the origin), so

$$F_r = -f(r) = -Cr^4$$  \hspace{1cm} (3.1)

(we are given that $C$ is positive) and the potential energy, taking $U(0) = 0$ is

$$U(r) = -\int_0^r F_r(r')dr' = \int_0^r f(r')dr' = C\int_0^r r'^4 dr' = \frac{C}{5}r^5.$$  \hspace{1cm} (3.2)

b) The effective potential is

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} + U(r) = \frac{L^2}{2\mu r^2} + \frac{C}{5}r^5$$ \hspace{1cm} (3.3)

The scales used in the graph below will be explained below. This graph is clearly computer-generated, and not really a “sketch,” but hey, this is MIT.
c) The circular orbit will correspond to the minimum effective potential; at this radius the kinetic energy will have no contribution from any radial motion. This minimum effective potential, and hence the radius of the circular orbit, is found from basic calculus and algebra,

\[
\left[ \frac{d}{dr} U_{\text{eff}}(r) \right]_{r=r_0} = -\frac{L^2}{\mu r_0^3} + C r_0^4
\]

\[
r_0^7 = \frac{L^2}{\mu C}, \quad r_0 = \left( \frac{L^2}{\mu C} \right)^{1/7}.
\]  

\[\text{(3.4)}\]

d) There are many, literally infinitely many correct answers since we are allowed to use \(r_0\), which have found in terms of \(L\), \(\mu\) and \(C\) in the answer. The answer presented here will not be in terms of \(r_0\). Hold on for some straightforward but unattractive algebra.

\[
E_{\text{min}} = U_{\text{eff}}(r_0) = \frac{L^2}{2\mu r_0^2} + \frac{C}{5}r_0^5
\]

\[
= \frac{L^2}{2\mu} \frac{\mu^{2/7} C^{2/7}}{L^{4/7}} + \frac{C}{5} \frac{L^{10/7}}{\mu^{5/7} C^{5/7}}
\]

\[
= \frac{L^{10/7} C^{2/7}}{\mu^{5/7}} \frac{7}{10} = \left[ \frac{L^9 C^2}{\mu^5} \right]^{7/10} \frac{7}{10}.
\]  

\[\text{(3.5)}\]
Here’s the advantage to doing this work. If we introduce the dimensionless ratio \( r = r / r_0 \), we can express the effective potential as

\[
U_{\text{eff}}(r) = \frac{L^2}{2 \mu r_0^2} \left( r_0^2 \frac{r}{r_0} \right)^2 + \frac{C r_0^5}{4} \left( \frac{r}{r_0} \right)^5
\]

\[
= E_{\text{min}} \left[ \frac{10}{7} \left( \frac{1}{2} \frac{1}{r} + \frac{1}{5} r^5 \right) \right].
\]

In the figure given for part (b), the vertical scale is in units of \( E_{\text{min}} \) and the horizontal scale is in units of \( r_0 \). The red curve (if viewed in color, the “upper” curve if not) is a graph of the quantity in square brackets in the second expression in (3.6). It is clear (I hope) from the figure that the minimum occurs at \( U_{\text{eff}} = E_{\text{min}} \) at \( r = r_0 \).

e) Since the quantity \( E \) is not (and in general cannot be) expressed in terms of the other parameters, there is no advantage in using the expression for \( E_{\text{min}} \) given in (3.5). Recall that the kinetic energy is

\[
K = \frac{L^2}{2 \mu r^2} + \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2.
\]

The difference \( E - E_{\text{min}} \) is then found by evaluating \( U_{\text{eff}} \) at \( r = r_0 \),

\[
E - E_{\text{min}} = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)_{r=r_0}^2,
\]

or \( |dr / dt| = \sqrt{\frac{2}{\mu} \left( E - E_{\text{min}} \right)} \).

f) No; \( dr / dt = 0 \), but the kinetic energy, from Equation (3.7), is

\[
K_{\text{min}} = \frac{L^2}{2 \mu r_{\text{max}}^2}.
\]
Problem 4 (10 points)

In a grain mill, grain is ground by a massive wheel which rolls without slipping in a circle on a flat horizontal surface driven by a vertical shaft. The rolling wheel has radius $b$ and is constrained to roll in a horizontal circle of radius $R$ at angular speed $\Omega$. Because of the stone’s angular momentum, the contact force with the surface can be considerably greater than the weight of the wheel. In this problem, the angular speed $\Omega$ about the shaft is such that the contact force between the ground and the wheel is equal to twice the weight. The goal of the problem is to find $\Omega$. Assume that the wheel is closely fitted to the axle so that it cannot tip, and that the width of the wheel $c \ll R$. Neglect friction and the mass of the axle of the wheel. Express your answer in terms of $R$, $b$, $M$, $\Omega$, and $g$ as needed.

a) How is the angular speed $\omega$ of the wheel about its axis related to the angular speed $\Omega$ about the shaft?

b) What is the horizontal component of the angular momentum vector about the point P in the figure above? Although we have not shown this, for this situation it is correct to compute the horizontal component of the angular momentum by completely ignoring the rotation of the mill wheel about the vertical axis, taking into account only the rotation of the mill wheel about its own axle.

c) Draw a free body force diagram for all the forces acting on the axle–wheel combination.

d) What is the torque about the joint (about the point $P$ in the figure above) due to the forces acting on the axle–wheel combination? Your answer may include any of the given variables $R$, $b$, $M$, $\Omega$ and $g$, and also any forces that you introduced in the force diagram of part (c).

e) Use the rotational equation of motion to find the value of $\Omega$ that doubles the contact force between the stone and the ground. Your final answer should be expressed in terms of $R$, $b$, $M$ and $g$, as needed.
Solutions:

First off, here’s a figure that shows the pivot point $P$ (omitted from early versions of the problem) but not the wheel width $c$, along with some convenient coordinate axes. Note that these cylindrical coordinates are not the same as those used in Problem 1(a).

![Diagram](image)

a) For rolling without slipping, any distance traced out on the ground must be the same as the distance a point on the wheel moves, and dividing out any time interval,

$$b\omega = R\Omega, \quad \omega = \frac{R}{b}. \quad (4.1)$$

b) The horizontal component of the angular momentum is the product of the angular velocity $\omega$ found in part (a), Equation (4.1), and the moment of inertia. Assuming a uniform millwheel, $I_{cm} = \frac{1}{2}Mb^2$, and so

$$L_{\text{horiz}} = \omega I_{cm} = \frac{1}{2} \omega Mb^2 = \frac{1}{2} \Omega MRb. \quad (4.2)$$

Although not asked in the problem, the horizontal component of $\mathbf{L}$ is directed inward, and in vector form $\mathbf{L}_{\text{horiz}} = L_{\text{horiz}} (-\hat{r})$ in the above coordinate system.

c) (Diagram follows part (d) below.) The axle exerts both a force and torque on the wheel, and this force and torque would be quite complicated. That’s why we consider the forces and torques on the axle/wheel combination. The pivot (or hinge) at point $P$ therefore must exert an inward vertical force to maintain the circular motion and an upward force to reflect that the upward normal force is larger in magnitude than the weight. The weight and the normal force in the limit $c << R$ would both act through the center of mass, but the normal force has been displaced for clarity. The relative magnitudes of the weight and the normal force have been indicated, but not that of the horizontal component of $\mathbf{F}_{\text{axle}}$ (which is not part of this problem, but is equal to $M\Omega^2 R$; we don’t know the ratio $\Omega^2 R / g$).
d) About point $P$, $\vec{F}_{\text{pivot}}$ exerts no torque, which is why we’re not concerned in this problem about its magnitude or what we call it. The normal force exerts a torque of magnitude $NR$, directed out of the page, or, in vector form, $\vec{\tau}_{\text{normal}} = -NR\hat{\theta}$. The weight exerts a torque of magnitude $MgR$, directed into the page, or, in vector form, $\vec{\tau}_{\text{weight}} = MgR\hat{\theta}$. From the problem statement, and anticipating part (e), if the magnitude of the normal force is double the weight, the net torque will be out of the page of the above figure.

It should be noted that if the magnitude of the vertical component of $\vec{F}_{\text{pivot}}$ is recognized as being the weight of the wheel, taking torques about the center of the wheel yields the same result.

\[ \vec{N} \]

\[ \vec{F}_{\text{pivot}} \]

\[ Mg \]

\[ \hat{\theta} \]

\[ (d / dt)(-\vec{r}) = -\hat{\theta} \]. This is consistent with the net torque being out of the page in the above figure.

The magnitude of the rate of change of the angular momentum is the magnitude of the horizontal component times the angular frequency, or $\tau_{\text{net}} = \Omega L_{\text{horiz}}$. Using the result of parts (c) and (d), with $N = Mg$, gives

\[
\left( N - Mg \right) R = \Omega L_{\text{horiz}}
\]

\[
MgR = \frac{1}{2} \Omega^2 MRb
\]

\[
\Omega^2 = 2 \frac{g}{b}.
\]

(4.3)
Problem 5 (10 points)

The equation for any orbit in an inverse square gravitational field is given by

\[ r = \frac{r_0}{1 - \varepsilon \cos \theta} \]  \hspace{1cm} (5.1)

where

\[ r_0 = \frac{L^2}{\mu G m_1 m_2}. \]  \hspace{1cm} (5.2)

In Equation (5.2), \( L \) is the angular momentum, \( \mu = m_1 m_2 / (m_1 + m_2) \) is the reduced mass, and \( \varepsilon \) is the eccentricity of the orbit. When \( 0 < \varepsilon < 1 \), the orbit is an ellipse (in the above figure, \( \varepsilon = 3/5 \)). The period, \( T \), depends only on the length of the major axis, \( A \), of the ellipse, which is given by

\[ A = \left( \frac{2T^2 G \left( m_1 + m_2 \right)}{\pi^2} \right)^{1/3} = \frac{2r_0}{1 - \varepsilon^2}. \]  \hspace{1cm} (5.3)

Halley’s Comet is in an elliptic orbit about the sun. The eccentricity of the orbit is \( \varepsilon = 0.967 \) and the period is \( T = 76 \) y. The mass of the sun is \( m_{\text{sun}} = 1.99 \times 10^{30} \) kg. The mass of Halley’s Comet is negligible compared to the sun.

a) Using this data, determine the distance of Halley’s Comet at closest approach \( r_p \) (perihelion) to the sun, and furthest distance \( r_a \) (aphelion) from the sun.
b) What is the speed $v_p$ of Halley’s Comet when it is closest to the sun?

**Solutions:**

Before diving into the numerical calculations, let’s do some preliminary math.

First, note that when $m_{\text{comet}} \ll m_{\text{sun}}$ (note that $m_{\text{comet}}$ is not given in the problem), $m_1 + m_2 \rightarrow m_{\text{sun}}$ and

$$\mu = \frac{m_{\text{sun}} m_{\text{comet}}}{m_{\text{sun}} + m_{\text{comet}}} = m_{\text{comet}} \frac{1}{1 + \frac{m_{\text{comet}}}{m_{\text{sun}}}} \rightarrow m_{\text{comet}}.$$  \hspace{1cm} (5.4)

Next, from Equation (5.1), we have

$$r_p = \frac{r_0}{1 + \epsilon}, \quad r_a = \frac{r_0}{1 - \epsilon}$$ \hspace{1cm} (5.5)

and combining with Equation (5.3) gives

$$r_p = \frac{A}{2} r_0 (1 - \epsilon), \quad r_a = \frac{A}{2} r_0 (1 + \epsilon).$$ \hspace{1cm} (5.6)

As a quick check, note that $r_p + r_a = A$, the major axis.

Next, anticipating part (b), we expect to use angular momentum considerations. At perihelion, the comet must be moving perpendicular to the vector from the sun to the comet, and so the magnitude of the angular momentum in terms of $v_p$, $r_p$ and $m_{\text{comet}}$ is

$$L = m_{\text{comet}} v_p r_p.$$ \hspace{1cm} (5.7)

Combining with Equation (5.2) (with $m_1 = m_{\text{sun}}$, $m_2 = m_{\text{comet}}$, or vice versa) and the simplification for $\mu$ as given in Equation (5.4) and solving for $v_p$ gives

$$v_p = \frac{L}{m_{\text{comet}} r_p} = \frac{\sqrt{Gr_0 m_{\text{sun}}^2 m_{\text{comet}}}}{m_{\text{comet}} r_p} = \frac{\sqrt{Gr_0 m_{\text{sun}}}}{r_p} = \sqrt{G \frac{(1 + \epsilon) m_{\text{sun}}}{r_p}}.$$ \hspace{1cm} (5.8)

The mass of the comet does indeed drop out of this problem if $m_{\text{comet}} \ll m_{\text{sun}}$.

It’s time to do the numbers. The calculations presented here were done by computer, keeping almost arbitrary precision in the intermediate calculations, and rounded to three
figures (even though the period is given to only two figures). For the Newtonian gravitational constant, \( G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \) was used.

a) Since the major axis \( A \) is used for determining both \( r_p \) and \( r_a \), find that quantity first. From Equation (5.3), with \( m_1 + m_2 = m_{\text{sun}} \),

\[
A = \left( \frac{2T^2 G m_{\text{sun}}}{\pi^2} \right)^{1/3} = \left( \frac{2(76 \times 3.16 \times 10^7 \text{ s} \cdot \text{y}^{-1})(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2})(1.99 \times 10^{30} \text{ kg})}{\pi^2} \right)^{1/3} = 5.37 \times 10^{12} \text{ m}
\]

from which Equation (5.6) gives

\[
r_p = 8.86 \times 10^{10} \text{ m}, \quad r_a = 5.28 \times 10^{12} \text{ m}.
\]

These results are roughly half and thirty times the earth-sun distance, respectively; \( r_p \) is roughly the distance from the sun to the ex-planet Pluto. If fact, the period of Halley’s Comet is roughly \( 1/\sqrt{8} \) the period of Pluto’s orbit, consistent with Equation (5.3).

A graph of the orbit is shown here:

The tiny dot (red, if viewed in color) represents the sun and is not to scale; a circle representing the sun to scale on this scale is too small to be seen (\( r_p > 100 R_{\text{sun}} \)).

b) Equation (5.8) then gives, with the result in Equation (5.10),

\[

\left. \begin{align*}
 v_p &= \sqrt{\frac{G(1+\varepsilon)m_{\text{sun}}}{r_p}} = 5.43 \times 10^4 \text{ m} \cdot \text{s}^{-1}, \\
 v_{\text{escape}} &= \frac{1}{2}(1+\varepsilon), \\
 \end{align*} \right\} \text{for the eccentricity of this orbit.}

This is essentially (but of course smaller than) the escape velocity from the sun. In fact it’s not too hard to show that \( v_p^2 / v_{\text{escape}}^2 = \frac{1}{2}(1+\varepsilon) \), which is 0.9835 for the eccentricity of this orbit.