A spaceship is sent to investigate a planet of mass $m_p$ and radius $r_p$. While hanging motionless in space at a distance $5r_p$ from the center of the planet, the ship fires an instrument package with speed $v_0$. The package has mass $m_i$ which is much smaller than the mass of the spacecraft. The package is launched at an angle $\theta$ with respect to a radial line between the center of the planet and the spacecraft. For what angle $\theta$ will the package just graze the surface of the planet?

**Solution:** The gravitational force by the planet on the spaceship $\vec{F}_{pm}^G$ always points towards the center of the planet. The torque about the center of the planet (point labeled $O$) due to the gravitational force is given by the expression

$$\vec{\tau}_O = \vec{r}_{O,m} \times \vec{F}_{pm}^G.$$  \hfill (1)

The vector $\vec{r}_{O,m}$ points from the center of the planet to the spaceship so $\vec{r}_{O,m}$ and $\vec{F}_{pm}^G$ are anti-parallel hence the torque $\vec{\tau}_O$ is zero. Therefore the angular momentum of the spaceship about the center of the planet is constant. In the figure below the several positions of the spaceship with the associated velocities are shown. (Think of this as an angular momentum diagram.) In the figure denote $v_0 \equiv v_{i,0}$.
Then the initial angular momentum of the spaceship about the center of the planet is

\[ \vec{L}_{0,i} = \vec{r}_{0,i} \times m_1 \vec{v}_{1,0} = 5 R \hat{r} \times (-m_0 \vec{v}_0 \cos \theta \hat{r} + m_1 \vec{v}_0 \sin \theta \hat{\theta}) = 5Rm_0 v_0 \sin \theta \hat{k}. \]  

The angular momentum about the center of the planet when the spaceship just grazes the planet is

\[ \vec{L}_{0,f} = \vec{r}_{0,f} \times m_1 \vec{v}_{1,f} = R m_1 v_{1,f} \hat{k}. \]  

Because angular momentum about the center of the planet is constant,

\[ \vec{L}_{0,i} = \vec{L}_{0,f}. \]  

Substituting Eqs. (2) and (3) into Eq. (4) and taking the z-component on both sides yields

\[ 5Rm_0 v_0 \sin \theta = Rm_1 v_{1,f}. \]  

Thus we can solve for the final speed

\[ v_{1,f} = 5v_0 \sin \theta. \]  

There is no non-conservative work done on the spacecraft so the mechanical energy is constant,

\[ E_i = E_f. \]  

Choose infinity as the zero point for potential energy then the energy equation becomes

\[ \frac{1}{2} m_1 v_0^2 - \frac{G m_0 m_p}{5R} = \frac{1}{2} m_1 v_{1,f}^2 - \frac{G m_1 m_p}{R}. \]  

Substitute in Eq. (6) for the final speed yielding
\[
\frac{1}{2} m_1 v_0^2 - \frac{G m_1 m_p}{5R} = \frac{1}{2} m_1 (5v_0 \sin \theta)^2 - \frac{G m_1 m_p}{R}.
\] (9)

Collecting terms we can rewrite Eq. (9) as

\[
\frac{4G m_1 m_p}{5R} = \frac{1}{2} m_1 v_0^2 (25 \sin^2 \theta - 1)
\] (10)

We can now solve for the angle \( \theta \):

\[
\theta = \sin^{-1} \left( \frac{1}{5} \sqrt{\frac{8G m_p}{5Rv_0^2 + 1}} \right).
\] (11)
IC-10D3-2 Group Problem Meteor Encounter Solution

A meteor of mass \( m \) is approaching earth as shown on the sketch. The distance \( h \) on the sketch below is called the impact parameter. The radius of the earth is \( R_e \). The mass of the earth is \( m_e \). Suppose the meteor has an initial speed of \( v_0 \). Assume that the meteor started very far away from the earth. Suppose the meteor just grazes the earth. You may ignore all other gravitational forces except the earth. Find the moment arm \( h \) (called the impact parameter).

Solution: As will be seen this problem is best done symbolically, with numerical values used at the end of the calculations. We’ll also need to neglect any air resistance when the meteor approaches the earth.

As the problem statement implies, we will need to conserve angular momentum. The meteor’s mass is so much small than the mass of the earth that we will assume that the earth’s motion is not affected by the meteor. Choose the center of the Earth, (point \( S \)) to calculate the torque and angular momentum. The force on the meteor is

\[
\mathbf{F}_{e,m}^G = -\frac{G m m_e r}{r^2} \mathbf{\hat{r}}.
\]  

(12)

The vector from the center of the Earth to the meteor is

\[
\mathbf{r}_S = r \mathbf{\hat{r}}.
\]

(13)

The torque about \( S \) is zero because they two vectors are anti-parallel

\[
\mathbf{\tau}_S = \mathbf{r}_S \times \mathbf{F}_{e,m}^G = r \mathbf{\hat{r}} \times -\frac{G m m_e}{r^2} \mathbf{\hat{r}} = 0.
\]

(14)

Therefore the angular momentum about the center of the earth is a constant. The initial angular momentum is
When the meteor just grazes the planet, the angular momentum is
\[ \mathbf{L}_{S,0} = \mathbf{r}_{S,0} \times m\mathbf{v}_0 = (x_0\hat{i} + h\hat{j}) \times m\mathbf{v}_0 \hat{i} = -hm\mathbf{k} . \] (15)

Therefore conservation of angular momentum requires that
\[ mv_0 h = mv_\alpha R_e . \] (17)

In order to solve for \( h \), we need to find \( v_\alpha \). Since we are neglecting all forces on the meteor other than the earth’s gravity, mechanical energy is conserved, and
\[ \frac{1}{2} mv_0^2 = \frac{1}{2} mv_\alpha^2 - \frac{Gmm_e}{R_e} \] (18)

where we have taken the meteor to have speed \( v_0 \) at a distance “very far away from the earth” to mean that we neglect any gravitational potential energy in the meteor-earth system. The parameter \( m \) is canceled from both Equations (17) and (18). Solving Equation (17) for \( v_\alpha \) and substitution into Equation (18) gives
\[ v_\alpha^2 = v_0^2 \left( \frac{h}{R_e} \right)^2 - \frac{2Gm_e}{R_e} . \] (19)

We can now solve for the parameter \( h \)
\[ h = \sqrt{R_e^2 + \frac{2Gm_e R_e}{v_0^2}} . \] (20)