Cartesian and Polar Coordinate Systems

Introduction

A coordinate system consists of four basic elements:

1) Choice of origin
2) Choice of axes
3) Choice of positive direction for each axis
4) Choice of unit vectors for each axis.

1. Cartesian Coordinates

Origin: Choose an origin \( O \). If you are given a geometric object, then your choice of origin may coincide with a special point in the body. For example, you may choose the mid-point of the wire.

Axes: Now we shall choose a set of axes. The simplest set of axes are known as the Cartesian axes, \( x \)-axis, \( y \)-axis, and the \( z \)-axis. In Figure 2.1.1, we draw these axes.

Then each point \( P \) in space our \( S \) can be assigned a triplet of values \( (x_P, y_P, z_P) \), the coordinates of the point \( P \). The ranges of values of the coordinates are: \( -\infty < x_p < +\infty, -\infty < y_p < +\infty, -\infty < z_p < +\infty \).

The collection of points that have the same the coordinate \( x_p \) is called a level surface. Suppose we ask what collection of points in our space \( S \) have the same value of \( x = x_p \). This is the set of points \( S_{x_p} = \{(x, y, z) \in S \text{ such that } x = x_p\} \). This set \( S_{x_p} \) is a plane, the \( y-z \) plane (Figure 2.1.2), called a level surface set for constant \( x_p \). So the \( x \)-coordinate of any point actually describes a plane of points perpendicular to the \( x \)-axis.
Figure 2.1.2 Level surface set for constant value $x_p$.

**Positive Direction:** Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol + along the positive axis. Conventionally, Cartesian coordinates are drawn with the $x$-$y$ plane corresponding to the plane of the paper. The horizontal direction from left to right is taken as the positive $x$-axis, and the vertical direction from bottom to top is taken as the positive $y$-axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes.

**Unit Vectors:** We now associate to each point $P$ in space, a set of three unit directions vectors ($\hat{i}_P, \hat{j}_P, \hat{k}_P$). A unit vector means has magnitude one; $|\hat{i}_P| = 1$, $|\hat{j}_P| = 1$, and $|\hat{k}_P| = 1$. We assign the direction of $\hat{i}_P$ to point in the direction of the increasing $x$-coordinate at the point $P$. We define the directions for $\hat{j}_P$ and $\hat{k}_P$ in the direction of the increasing $y$-coordinate and $z$-coordinate respectively, (Figure 2.1.3).
**Infinitesimal Line Elements:** Consider a small infinitesimal displacement $d\hat{s}$ between two points $P_1$ and $P_2$ (Figure 2.1.3a). This vector can be decomposed into

$$d\hat{s} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$  \hspace{1cm} (2.1.1)

![Figure 2.1.3a Displacement vector $d\hat{s}$ between two points](image)

**Infinitesimal Area Element:** An infinitesimal area element (Figure 2.1.4) of a thin sheet lying on the $xy$-plane is given by

$$dA = dx \, dy$$  \hspace{1cm} (2.1.2)

![Figure 2.1.4 Area element for a sheet](image)
\[ d\vec{A} = dx\, dy\, \hat{k} \]  \hspace{1cm} (2.1.3)

**Infinitesimal Volume Element:** An infinitesimal volume element (Figure 2.1.5) in Cartesian coordinates is given by

\[ dV = dx\, dy\, dz \]  \hspace{1cm} (2.1.4)

![Figure 2.1.5 Volume element in Cartesian coordinates](image)
2. Cylindrical Coordinates

We first choose an origin and an axis we call the $z$-axis with unit vector $\hat{z}$ pointing in the increasing $z$-direction. The level surface of points such that $z = z_{P}$ define a plane. We shall choose coordinates for a point $P$ in the plane $z = z_{P}$ as follows.

One coordinate, $r$, measures the distance from the $z$-axis to the point $P$. The coordinate $r$ ranges in value from $0 \leq r \leq \infty$. In Figure 2.2.1 we draw a few surfaces that have constant values of $r$. These 'level surfaces' are circles.

![Figure 2.2.1 level surfaces for the coordinate $r$](image)

Our second coordinate measures an angular distance along the circle. We need to choose some reference point to define the angle coordinate. We choose a 'reference ray', a horizontal ray starting from the origin and extending to $+\infty$ along the horizontal direction to the right. (In a typical Cartesian coordinate system, our 'reference ray' is the positive x-direction). We define the angle coordinate for the point $P$ as follows. We draw a ray from the origin to the point $P$. We define the angle $\theta$ as the angle in the counterclockwise direction between our horizontal reference ray and the ray from the origin to the point $P$, (see Figure 2.2.2):

![Figure 2.2.2 the angle coordinate](image)
All the other points that lie on a ray from the origin to infinity passing through $P$ have the same value as $\theta$. For any arbitrary point, our angle coordinate $\theta$ can take on values from $0 \leq \theta < 2\pi$. In Figure 2.2.3 we depict other ‘level surfaces’ which are lines in the plane for the angle coordinate. The coordinates $(r, \theta)$ in the plane $z = z_p$ are called polar coordinates.

![Figure 2.2.3 Level surfaces for the angle coordinate](image)

**Figure 2.2.3** Level surfaces for the angle coordinate

**Unit Vectors:** We choose two unit vectors in the plane at the point $P$ as follows. We choose $\hat{r}$ to point in the direction of increasing $r$, radially away from the $z$-axis. We choose $\hat{\theta}$ to point in the direction of increasing $\theta$. This unit vector points in the counterclockwise direction, tangent to the circle. Our complete coordinate system is shown in Figure 2.2.4. This coordinate system is called a ‘cylindrical coordinate system’. Essentially we have chosen two directions, radial and tangential in the plane and a perpendicular direction to the plane.

![Figure 2.2.4 Cylindrical coordinates](image)

**Figure 2.2.4** Cylindrical coordinates
If you are given polar coordinates \((r, \theta)\) of a point in the plane, the Cartesian coordinates \((x, y)\) can be determined from the coordinate transformations

\[
x = r \cos \theta \\
y = r \sin \theta
\] (2.2.1)

Conversely, if you are given the Cartesian coordinates \((x, y)\), the polar coordinates \((r, \theta)\) can be determined from the coordinate transformations

\[
r = +\sqrt{x^2 + y^2} \\
\theta = \tan^{-1}(y/x)
\] (2.2.3)

Note that \(r \geq 0\) so you always need to take the positive square root. Note also that \(\tan \theta = \tan(\theta + \pi)\). Suppose that \(0 \leq \theta \leq \pi/2\), then \(x \geq 0\) and \(y \geq 0\). Then the point \((-x, -y)\) will correspond to the angle \(\theta + \pi\).

The unit vectors also are related by the coordinate transformations

\[
\hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j} \\
\hat{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}
\] (2.2.5)

Similarly

\[
\hat{i} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \\
\hat{j} = \sin \theta \hat{r} + \cos \theta \hat{\theta}
\] (2.2.6)

One crucial difference between polar coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point \(S\) in the plane. The unit vectors in Cartesian coordinates \((\hat{i}_S, \hat{j}_S)\) at the point \(S\) have the same magnitude and point in the same direction as the unit vectors \((\hat{i}_P, \hat{j}_P)\) at \(P\). Any two vectors that are equal in magnitude and point in the same direction are equal; therefore

\[
\hat{i}_S = \hat{i}_P, \quad \hat{j}_S = \hat{j}_P
\] (2.2.9)

A Cartesian coordinate system is the unique coordinate system in which the set of unit vectors at different points in space are equal. In polar coordinates, the unit vectors at two different points are not equal because they point in different directions. We show this in Figure 2.2.5.
**Infinitesimal Line Elements:** Consider a small infinitesimal displacement $d\vec{s}$ between two points $P_1$ and $P_2$ (Figure 2.2.6). This vector can be decomposed into

$$d\vec{s} = dr\hat{r} + rd\theta\hat{\theta} + dz\hat{k} \quad (2.2.10)$$

**Infinitesimal Area Element:**

Consider an infinitesimal area element on the surface of a cylinder of radius $r$ (Figure 2.2.7).
The area of this element has magnitude

\[ dA = rd\theta dz \] (2.2.11)

Area elements are actually vectors where the direction of the vector \( d\hat{A} \) points perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface. So for the surface of the cylinder, the infinitesimal area vector is

\[ d\hat{A} = rd\theta dz \hat{r} \] (2.2.12)

Consider an infinitesimal area element on the surface of a disc (Figure 2.2.8) in the \( x\)-\( y \) plane.

**Figure 2.2.7** Area element for a cylinder

**Figure 2.2.8** Area element for a disc
This area element is given by the vector

\[ d\vec{A} = rd\theta dr \hat{k} \]  

(2.2.13)

**Infinitesimal volume element:**

An infinitesimal volume element (Figure 2.2.9) is given by

\[ dV = rd\theta dr dz \]  

(2.2.14)

**Figure 2.2.9** Volume element