Newton’s First Law

The First Law of Motion, commonly called the “Principle of Inertia,” was first realized by Galileo. (Newton did not acknowledge Galileo’s contribution.) Newton was particularly concerned with how to phrase the First Law in Latin, but after many rewrites Newton perfected the following expression for the First Law (in English translation):

Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Projectiles continue in their motions, so far as they are not retarded by the resistance of air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by air. The greater bodies of planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

The first law is an experimental statement about the motions of bodies. When a body moves with constant velocity, there are either no forces present or there are forces acting in opposite directions that cancel out. If the body changes its velocity, then there must be an acceleration, and hence a non-zero force must be present. We note that velocity can change in two ways. The first way is to change the magnitude of the velocity; the second way is to change its direction.

After a bus or train starts, the acceleration is often so small we can barely perceive it. We are often startled because it seems as if the station is moving in the opposite direction while we seem to be still. Newton’s First Law states that there is no physical way to distinguish between whether we are moving or the station is, because there is essentially no total force present to change the state of motion. Once we reach a constant velocity, our minds dismiss the idea that the ground is moving backwards because we think it is impossible, but there is no actual way for us to distinguish whether the train is moving or the ground is moving.

Reference Frames

In order to describe physical events that occur in space and time such as the motion of bodies, we introduced a coordinate system. A space-time event can now be specified by its spatial and temporal coordinates. In particular, the position of a moving body can be described by space-time events specified by the space-time coordinates. You can place an observer at the origin of coordinate system. The coordinate system with your observer
acts as a reference frame for describing the position, velocity, and acceleration of bodies. The position vector of the body depends on the choice of origin (location of your observer) but the displacement, velocity, and acceleration vectors are independent of the location of the observer.

You can always choose a second reference frame that is moving with respect to the first reference frame. Then the position, velocity and acceleration of bodies as seen by the different observers do depend on the relative motion of the two reference frames. The relative motion can be described in terms of the relative position, velocity, and acceleration of the observer at the origin, \( O \), in reference frame \( S \) with respect to a second observer located at the origin, \( O' \), in reference frame \( S' \).

Let the vector \( \vec{R} \) point from the origin of frame \( S \) to the origin of reference frame \( S' \). Suppose an object is located at a point \( 1 \). Denote the position vector of the object with respect to origin of reference frame \( S \) by \( \vec{r} \). Denote the position vector of the object with respect to origin of reference frame \( S' \) by \( \vec{r}' \) (Figure 1).

![Two reference frames.](image)

**Figure 1** Two reference frames.

The position vectors are related by

\[
\vec{r}' = \vec{r} - \vec{R}
\]  

These coordinate transformations are called the **Galilean Coordinate Transformations**. They enable the observer in frame \( S \) to predict the position vector in frame \( S' \), based only on the position vector in frame \( S \) and the relative position of the origins of the two frames.

The relative velocity between the two reference frames is given by the time derivative of the vector \( \vec{R} \), defined as the limit as of the displacement of the two origins divided by an interval of time, as the interval of time becomes infinitesimally small,

\[
\vec{V} = \frac{d\vec{R}}{dt}
\]
Relatively Inertial Reference Frames

If the relative velocity between the two reference frames is constant, then the relative acceleration between the two reference frames is zero,

$$\vec{A} = \frac{d\vec{V}}{dt} = \vec{0}. \quad (3)$$

When two reference frames are moving with a constant velocity relative to each other as above, the reference frames are considered to be relatively inertial reference frames. We can reinterpret Newton’s First Law

Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

as the Principle of Relativity:

In relatively inertial reference frames, if there is no net force impressed on an object at rest in frame $S$, then there is also no net force impressed on the object in frame $S'$.

Law of Addition of Velocities: Newtonian Mechanics

Suppose the object in Figure 1 is moving; then observers in different reference frames will in general measure different velocities. Denote the velocity of the object in frame $S$ by $\vec{v} = d\vec{r}/dt$, and the velocity of the object in frame $S'$ by $\vec{v}' = d\vec{r}'/dt'$. Since the derivative of the position is velocity, the velocities of the object in two different reference frames are related according to

$$\frac{d\vec{r}'}{dt'} = \frac{d\vec{r}}{dt} - \frac{d\vec{R}}{dt} \quad (4)$$

or

$$\vec{v}' = \vec{v} - \vec{V} \quad (5)$$

This is called the Law of Addition of Velocities.
Problems

4.1 Relative Velocities of Two Moving Cars

Suppose two cars, Car A and Car B, are traveling along roads that are perpendicular to each other (Figure 2). An observer at rest with respect to the ground defines reference frame $S'$. According to this observer, Car A is moving with velocity $\mathbf{v}_A = v_A \mathbf{j}$, and Car B is moving with velocity $\mathbf{v}_B = v_B \mathbf{i}$.

Consider a second observer moving along with Car A, defining reference frame $S''$. What is the velocity of Car B according to this observer moving in Car A? The velocity of the observer moving along in Car A with respect to an observer at rest on the ground is just the velocity of Car A and is given by $\mathbf{V} = \mathbf{v}_A = v_A \mathbf{j}$. Using the Law of Addition of Velocities, Equation (5), the velocity of Car B with respect to an observer moving along with Car A is given by

$$\mathbf{v}_B' = \mathbf{v}_B - \mathbf{V} = v_B \mathbf{i} - v_A \mathbf{j}. \tag{6}$$

We can now use Equation (2.3.5) to find the magnitude of velocity of Car B as seen by an observer moving with Car A,

$$|\mathbf{v}_B'| = \left( v_B^2 + v_A^2 \right)^{1/2}. \tag{7}$$
4.2 Rowing Across the River

An MIT student wants to row across the Charles River. Suppose the water is moving downstream at a constant rate of 1.0 m/s. A second boat is floating downstream with the current. From the second boat’s viewpoint, the student is rowing perpendicular to the current at 0.5 m/s. Suppose the river is 800 m wide.

a) What is the direction and magnitude of the velocity of the student as seen from an observer at rest along the bank of the river?

b) How far down river does the student land on the opposite bank?

c) How long does the student take to reach the other side?

Solution

Choose a coordinate system with the \( \hat{i} \)-direction across the river (more or less south) and the \( \hat{j} \)-direction downstream (more or less east). Denote the velocity vector of the boat moving with the current, as observed from the riverbank, as \( \vec{V} = v_A \hat{j} \) and the velocity of the rower, as observed from the riverbank, as \( \vec{v}_B \). The velocity of the rower as observed by the floating boat is

\[
\vec{v}_B' = \vec{v}_B - \vec{V} = \vec{v}_B - v_A \hat{j}.
\] (8)

a) Solving (8) for \( \vec{v}_B' \), \( \vec{v}_B = \vec{v}_B' + \vec{V} \). We are given that \( \vec{v}_B' \perp \vec{V} \);

\[
\vec{v}_B = v_B' \hat{j} + v_A \hat{i} = (0.5 \text{ m/s}) \hat{j} + (1.0 \text{ m/s}) \hat{i}.
\] (9)

The magnitude and direction of \( \vec{v}_B \) are

\[
v_B = \sqrt{(0.5 \text{ m/s})^2 + (1.0 \text{ m/s})^2} = 1.1 \text{ m/s}
\] (10)

to two significant figures and

\[
\theta_B = \tan^{-1}\left(\frac{v_A}{v_B}\right) = \tan^{-1}\left(\frac{1.0 \text{ m/s}}{0.5 \text{ m/s}}\right) = 64^\circ.
\] (11)
While the answer in (11) is to two significant figures, this is probably far more precise than is warranted. Rounding to the nearest five or even ten degrees would be more appropriate.
4.3 Pebble Lodged in a Tire

A bicycle wheel of radius $a$ is rolling in a straight line without slipping at a constant horizontal speed $V$. A bead is fixed to a spoke a distance $b$ from the center of the wheel.

a) Find the position and velocity of the bead as a function of time as seen by an observer located at the center of the wheel and moving with the wheel. Make sure you use appropriate unit vectors in your answer.

b) What is the position and velocity of the observer at the center of the wheel as seen by an observer fixed to the ground. Assume at $t = 0$ that the center of the wheel is directly over the observer fixed to the ground. Make sure you use appropriate unit vectors in your answer.

c) What is the relation between the angular speed of the wheel, $\omega$, and the horizontal speed, $V$, of the wheel?

d) Find the position and velocity of the bead as a function of time as seen by the observer fixed to the ground. Make sure you use appropriate unit vectors in your answer.

Solution:

a) Choose a reference frame with an origin at the center of the wheel, and moving with the wheel. Choose polar coordinates. The angular speed is $\omega = d\theta / dt$.

Then the bead is undergoing uniform circular motion with the position, velocity, and acceleration given by
\[ \mathbf{r}' = b \mathbf{r} \quad \mathbf{v}' = b\omega \mathbf{\dot{\theta}} \quad \mathbf{a}' = -b\omega^2 \mathbf{r} \]

Rolling without slipping:

Because the wheel is rolling without slipping, the velocity of a point on the rim of the wheel has speed \( v_{\text{rim}} = a\omega \). This is equal to the linear speed of the center of mass of the wheel \( V_{\text{cm}} = V \), thus

\[ a\omega = V \quad \text{or} \quad \omega = V / a \]

Note that the at \( t = 0 \), the angle \( \theta = \theta_0 = 0 \). So the angle grows in time as

\[ \theta = \omega t = (V / a)t \].

So the velocity and acceleration of the bead with respect to the center of the wheel become

\[ \mathbf{v}' = (b / a)V \mathbf{\dot{\theta}} \quad \mathbf{a}' = -(b / a^2)V^2 \mathbf{\hat{r}} \]

b) Define a second reference frame fixed to the ground with choice of origin, Cartesian coordinates and unit vectors as shown in the figure below.

Then the relative position vector of the moving origin of the frame in part (a) to the origin in the second frame is given by

\[ \mathbf{R} = X \mathbf{\hat{i}} + a \mathbf{\hat{j}}. \]

The relative velocity of the two frames is the derivative

\[ \mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{dX}{dt} \mathbf{\hat{i}} = V \mathbf{\hat{i}}. \]
Since the center of the wheel is moving at a uniform speed the relative acceleration of the two frames is zero,

\[ \ddot{A} = \frac{d\ddot{V}}{dt} = 0 \]

Note that the at \( t = 0 \), the angle \( \theta = \theta_0 = 0 \). So the angle grows in time as

\[ \theta = \omega t . \]

Define the position, velocity, and acceleration in this frame (with respect to the ground) by

\[ \hat{r} = x \hat{i} + y \hat{j} \quad \hat{v} = v_x \hat{i} + v_y \hat{j} \quad \hat{a} = a_x \hat{i} + a_y \hat{j} \]

Then the position vectors are related by

\[ \vec{r} = \hat{R} + \vec{r}' . \]

In order to add these vectors we need to decompose the position vector in the moving frame into Cartesian components,

\[ \hat{r}' = b \hat{r} = b \sin \theta \hat{i} + b \cos \theta \hat{j} . \]

Then

\[ \hat{r} = \hat{R} + \hat{r}' = (X \hat{i} + a \hat{j}) + (b \sin \theta \hat{i} + b \cos \theta \hat{j}) = (X \hat{i} + b \sin \theta \hat{i}) + (a + b \cos \theta \hat{j}) . \]

Thus the position components of the bead with respect to the ground are given by

\[ x = X + b \sin((V / a)t) \]
\[ y = a + b \cos((V / a)t) \]

We can differentiate the position vector to find the velocity

\[ \ddot{v} = \frac{d\ddot{r}}{dt} = \frac{d}{dt} \left( X + b \sin((V / a)t) \right) \hat{i} + \frac{d}{dt} \left( a + b \cos((V / a)t) \right) \hat{j} \]

\[ \ddot{v} = (V + (b / a)V \cos((V / a)t)) \hat{i} - ((b / a)V \sin((V / a)t)) \hat{j} \]
Alternatively, we can decompose the velocity of the bead in the moving frame into Cartesian coordinates

\[ \mathbf{v}' = \left( \frac{b}{a} \right) V \left( \cos((V/a)t) \mathbf{i} - \sin((V/a)t) \mathbf{j} \right) \]

Then velocities are related by the law of addition of velocities

\[ \mathbf{v} = \mathbf{v} + \mathbf{v}' \]

so

\[ \mathbf{v} = V \mathbf{i} + \left( \frac{b}{a} \right) V \left( \cos((V/a)t) \mathbf{i} - \sin((V/a)t) \mathbf{j} \right) \]

\[ \mathbf{v} = \left( V + \left( \frac{b}{a} \right) V \cos((V/a)t) \right) \mathbf{i} - \left( \frac{b}{a} \right) \sin((V/a)t) \mathbf{j} \]

in agreement with our previous result.

The acceleration is the same in either frame so

\[ \mathbf{a} = \mathbf{a}' = -\left( \frac{b}{a^2} \right) V^2 \mathbf{r} = -\left( \frac{b}{a^2} \right) V^2 \left( \sin((V/a)t) \mathbf{i} + \cos((V/a)t) \mathbf{j} \right) \]
4.4 Sliding on a Ring

By relative velocity we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer’s coordinate system.)

a. A point is observed to have velocity $\mathbf{v}_A$ relative to coordinate system $A$. What is its velocity relative to coordinate system $B$, which is displaced from system $A$ by distance $\mathbf{R}$? ($\mathbf{R}$ can change in time.)

b. Particles $a$ and $b$ move in opposite directions around a circle with angular speed $\omega$, as shown. At $t = 0$ they are both at the point $\mathbf{r} = l\mathbf{j}$, where $l$ is the radius of the circle. Find the velocity of $a$ relative to $b$.

Solution: (a) The position vectors are related by

$$\mathbf{r}_B = \mathbf{r}_A - \mathbf{R}. \quad (12)$$

Then velocities are related by the taking derivatives, (law of addition of velocities)

$$\mathbf{v}_B = \mathbf{v}_A - \mathbf{\Omega}. \quad (13)$$

(b) Let’s choose two reference frames; frame $B$ is centered at particle $b$, and frame $A$ is centered at the center of the circle in the figure below.
Then the relative position vector between the origins of the two frames is given by

$$\mathbf{\bar{R}} = l \mathbf{\hat{r}}.$$  \hspace{1cm} (14)

The position vector of particle a relative to frame A is given by

$$\mathbf{\bar{r}}_{a} = l \mathbf{\hat{r}}.$$

The position vector of particle b in frame B can be found by substituting Eqs. (15) and (14) into Eq. (12),

$$\mathbf{\bar{r}}_{b} = \mathbf{\bar{r}}_{a} - \mathbf{\bar{R}} = l \mathbf{\hat{r}} - l \mathbf{\hat{r}}'. \hspace{1cm} (16)$$

We can decompose each of the unit vectors $\mathbf{\hat{r}}$ and $\mathbf{\hat{r}}'$ with respect to the Cartesian unit vectors $\mathbf{\hat{i}}$ and $\mathbf{\hat{j}}$ (see figure)

$$\mathbf{\hat{r}} = -\sin \theta \mathbf{\hat{i}} + \cos \theta \mathbf{\hat{j}} \hspace{1cm} (17)$$
$$\mathbf{\hat{r}}' = \sin \theta \mathbf{\hat{i}} + \cos \theta \mathbf{\hat{j}}. \hspace{1cm} (18)$$

Then Eq. (16) giving the position vector of particle b in frame B becomes

$$\mathbf{\bar{r}}_{b} = l \mathbf{\hat{r}}' - l \mathbf{\hat{r}} = l (\sin \theta \mathbf{\hat{i}} + \cos \theta \mathbf{\hat{j}}) - l (\sin \theta \mathbf{\hat{i}} + \cos \theta \mathbf{\hat{j}}) = 2l \sin \theta \mathbf{\hat{i}}. \hspace{1cm} (19)$$

In order to find the velocity vector of particle a in frame B (i.e. with respect to particle b), differentiate Eq. (19)

$$\mathbf{\bar{v}}_{b} = \frac{d}{dt} (2l \sin \theta) \mathbf{\hat{i}} = (2l \cos \theta) \frac{d\theta}{dt} \mathbf{\hat{i}} = 2.0l \cos \theta \mathbf{\hat{i}}. \hspace{1cm} (20)$$