Math Review 3: Circular Motion

Introduction

Special cases often dominate our study of physics, and circular motion is certainly no exception. We see circular motion in many instances in the world; a bicycle rider on a circular track, a ball spun around by a string, and the rotation of a spinning wheel are just a few examples. Various planetary models described the motion of planets in circles before any understanding of gravitation. The motion of the moon around the earth is nearly circular. The motions of the planets around the sun are nearly circular. Our sun moves in nearly a circular orbit about the center of our galaxy, 50,000 light years from a massive black hole at the center of the galaxy.

We shall describe the kinematics of circular motion, the position, velocity, and acceleration, as a special case of two-dimensional motion. We will see that unlike linear motion, where velocity and acceleration are directed along the line of motion, in circular motion the direction of velocity is always tangent to the circle. This means that as the object moves in a circle, the direction of the velocity is always changing. When we examine this motion, we shall see that the direction of change of the velocity is towards the center of the circle. This means that there is a non-zero component of the acceleration directed radially inward, which is called the centripetal acceleration. If our object is increasing its speed or slowing down, there is also a non-zero tangential acceleration in the direction of motion. But when the object is moving at a constant speed in a circle then only the centripetal acceleration is non-zero.

In all of these instances, when an object is constrained to move in a circle, there must exist a force \( \vec{F} \) acting on the object directed towards the center.

In 1666, twenty years before Newton published his *Principia*, he realized that the moon is always “falling” towards the center of the earth; otherwise, by the First Law, it would continue in some linear trajectory rather than follow a circular orbit. Therefore there must be a centripetal force, a radial force pointing inward, producing this centripetal acceleration.

Since Newton’s Second Law \( \vec{F} = m\vec{a} \) is a vector equality, it can be applied to the radial direction to yield

\[
F_{\text{radial}} = m a_{\text{radial}}. \tag{3.1.1}
\]
3.1 Cylindrical Coordinate System

We first choose an origin and an axis we call the $z$-axis with unit vector $\hat{z}$ pointing in the increasing $z$-direction. The level surface of points such that $z = z_P$ define a plane. We shall choose coordinates for a point $P$ in the plane $z = z_P$ as follows.

One coordinate, $r$, measures the distance from the $z$-axis to the point $P$. The coordinate $r$ ranges in value from $0 \leq r \leq \infty$. In Figure 3.1.1 we draw a few surfaces that have constant values of $r$. These ‘level surfaces’ are circles.

![Figure 3.1.1 level surfaces for the coordinate $r$](image)

Our second coordinate measures an angular distance along the circle. We need to choose some reference point to define the angle coordinate. We choose a ‘reference ray’, a horizontal ray starting from the origin and extending to $+\infty$ along the horizontal direction to the right. (In a typical Cartesian coordinate system, our ‘reference ray’ is the positive $x$-direction). We define the angle coordinate for the point $P$ as follows. We draw a ray from the origin to the point $P$. We define the angle $\theta$ as the angle in the counterclockwise direction between our horizontal reference ray and the ray from the origin to the point $P$, (see Figure 3.1.2):

![Figure 3.1.2 the angle coordinate](image)
All the other points that lie on a ray from the origin to infinity passing through $P$ have the same value as $\theta$. For any arbitrary point, our angle coordinate $\theta$ can take on values from $0 \leq \theta < 2\pi$. In Figure 3.1.3 we depict other ‘level surfaces’ which are lines in the plane for the angle coordinate. The coordinates $(r,\theta)$ in the plane $z = z_P$ are called polar coordinates.

![Figure 3.1.3 Level surfaces for the angle coordinate](image)

**Unit Vectors:** We choose two unit vectors in the plane at the point $P$ as follows. We choose $\hat{r}$ to point in the direction of increasing $r$, radially away from the $z$-axis. We choose $\hat{\theta}$ to point in the direction of increasing $\theta$. This unit vector points in the counterclockwise direction, tangent to the circle. Our complete coordinate system is shown in Figure 3.1.4. This coordinate system is called a ‘cylindrical coordinate system’. Essentially we have chosen two directions, radial and tangential in the plane and a perpendicular direction to the plane.

![Figure 3.1.4 Cylindrical coordinates](image)
If you are given polar coordinates \((r, \theta)\) of a point in the plane, the Cartesian coordinates \((x, y)\) can be determined from the coordinate transformations

\[
x = r \cos \theta \\
y = r \sin \theta
\]  

(3.1.2) \hspace{1cm} (3.1.3)

Conversely, if you are given the Cartesian coordinates \((x, y)\), the polar coordinates \((r, \theta)\) can be determined from the coordinate transformations

\[
r = \pm \sqrt{x^2 + y^2} \]  

(3.1.4)

\[
\theta = \tan^{-1}(y/x)
\]  

(3.1.5)

Note that \(r \geq 0\) so you always need to take the positive square root. Note also that \(\tan \theta = \tan(\theta + \pi)\). Suppose that \(0 \leq \theta \leq \pi/2\), then \(x \geq 0\) and \(y \geq 0\). Then the point \((-x, -y)\) will correspond to the angle \(\theta + \pi\).

The unit vectors also are related by the coordinate transformations

\[
\hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j} \\
\hat{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}
\]  

(3.1.6) \hspace{1cm} (3.1.7)

Similarly

\[
\hat{i} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \\
\hat{j} = \sin \theta \hat{r} + \cos \theta \hat{\theta}
\]  

(3.1.8) \hspace{1cm} (3.1.9)

One crucial difference between polar coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point \(S\) in the plane. The unit vectors in Cartesian coordinates \((\hat{i}_S, \hat{j}_S)\) at the point \(S\) have the same magnitude and point in the same direction as the unit vectors \((\hat{i}_P, \hat{j}_P)\) at \(P\). Any two vectors that are equal in magnitude and point in the same direction are equal; therefore

\[\hat{i}_S = \hat{i}_P, \quad \hat{j}_S = \hat{j}_P\]  

(3.1.10)

A Cartesian coordinate system is the unique coordinate system in which the set of unit vectors at different points in space are equal. In polar coordinates, the unit vectors at two different points are not equal because they point in different directions. We show this in Figure 3.1.5.
Infinitesimal Line Elements: Consider a small infinitesimal displacement $d\vec{s}$ between two points $P_1$ and $P_2$ (Figure 3.1.6). This vector can be decomposed into

$$d\vec{s} = dr \hat{r} + rd\theta \hat{\theta} + dz \hat{k}$$  \hspace{1cm} (3.1.11)

Infinitesimal Area Element:

Consider an infinitesimal area element on the surface of a cylinder of radius $r$ (Figure 3.1.7).
The area of this element has magnitude

\[ dA = r d\theta dz \]  \hspace{1cm} (3.1.12)

Area elements are actually vectors where the direction of the vector \( d\vec{A} \) points perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface. So for the surface of the cylinder, the infinitesimal area vector is

\[ d\vec{A} = rd\theta dz \hat{r} \]  \hspace{1cm} (3.1.13)

Consider an infinitesimal area element on the surface of a disc (Figure 3.1.8) in the \( x\)-\( y \) plane.
This area element is given by the vector

\[ d\mathbf{A} = r d\theta dr \mathbf{\hat{k}} \]  \hspace{1cm} (3.1.14)

**Infinitesimal volume element:**

An infinitesimal volume element (Figure 3.1.9) is given by

\[ dV = r d\theta dr dz \]  \hspace{1cm} (3.1.15)

![Volume element](image)

**Figure 3.1.9 Volume element**

The motion of objects moving in circles motivates the use of the cylindrical coordinate system. This is ideal, as the mathematical description of this motion makes use of the radial symmetry of the motion. Consider the central radial point and a vertical axis passing perpendicular to the plane of motion passing through that central point. Then any rotation about this vertical axis leaves circles invariant (unchanged), making this system ideal for use for analysis of circular motion exploiting of the radial symmetry of the motion.
3.2 Circular Motion: Velocity and Angular Velocity

We can now begin our description of circular motion. In Figure 3.2.1 we sketch the position vector \( \vec{r}(t) \) of the object moving in a circular orbit of radius \( R \).

\[
\vec{r}(t) = R\hat{r}
\]  

(3.2.1)

![Figure 3.2.1 A circular orbit.](image)

The magnitude of the displacement, \( |\Delta \vec{r}| \), is the represented by the length of the horizontal vector \( \Delta \vec{r} \) joining the heads of the displacement vectors in Figure 3.2.2 and is given by

\[
|\Delta \vec{r}| = 2R \sin(\Delta \theta / 2)
\]

(3.2.2)

![Figure 3.2.2 Change in position vector for circular motion.](image)

When the angle \( \Delta \theta \) is small, we can approximate

\[
\sin(\Delta \theta / 2) \approx \Delta \theta / 2.
\]

(3.2.3)
This is called the *small angle approximation*, where the angle $\Delta \theta$ (and hence $\Delta \theta / 2$) is measured in radians. This fact follows from an infinite power series expansion for the sine function given by

$$\sin\left(\frac{\Delta \theta}{2}\right) = \frac{\Delta \theta}{2} - \frac{1}{3!}\left(\frac{\Delta \theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\Delta \theta}{2}\right)^5 - \cdots. \quad (3.2.4)$$

When the angle $\Delta \theta / 2$ is small, only the first term in the infinite series contributes, as successive terms in the expansion become much smaller. For example, when $\Delta \theta / 2 = \pi / 30 \equiv 0.1$, corresponding to $6^\circ$, $(\Delta \theta / 2)^3 / 3! \approx 1.9 \times 10^{-4}$; this term in the power series is three orders of magnitude smaller than the first and can be safely ignored for small angles.

Using the small angle approximation, the magnitude of the displacement is

$$|\Delta \vec{r}| \equiv R \Delta \theta. \quad (3.2.5)$$

This result should not be too surprising since in the limit as $\Delta \theta$ approaches zero, the length of the chord approaches the arc length $R \Delta \theta$.

The magnitude of the velocity, $|\vec{v}| \equiv v$, is then seen to be proportional to the rate of change of the magnitude of the angle with respect to time,

$$v \equiv |\vec{v}| = \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{R |\Delta \theta|}{\Delta t} = R \lim_{\Delta t \to 0} \frac{|\Delta \theta|}{\Delta t} = R \left|\frac{d\theta}{dt}\right|. \quad (3.2.6)$$

**Definition: Angular Velocity**

The rate of change of angle with respect to time is called the *angular velocity* and is denoted by the Greek letter $\omega$,

$$\omega \equiv \frac{d\theta}{dt}. \quad (3.2.7)$$

The SI units of angular velocity are $[\text{rad} \cdot \text{s}^{-1}]$.

Thus the magnitude of the velocity for circular motion is given by

$$v = R |\omega|. \quad (3.2.8)$$

The direction of the velocity can be determined by considering that in Figure 3.3, with $|\vec{r}(t)| = |\Delta \vec{r}(t + \Delta t)| = R$ a constant, the two position vectors put tail-to-tail form two sides
of an isosceles triangle, and the third side, corresponding to $\Delta \mathbf{r}$, must be perpendicular to the bisector of the triangle, regardless of the value of $\Delta \theta$; thus, in the limit $\Delta \theta \to 0$, $\Delta \mathbf{r} \perp \mathbf{r}$, and so the direction of the velocity $\mathbf{v}$ at time $t$ is perpendicular to position vector $\mathbf{r}(t)$ and tangent to the circular orbit in the $\hat{\theta}$-direction.

There’s a rigorous way to show this mathematically: Consider that since $R^2 = \mathbf{r} \cdot \mathbf{r}$ is a constant,
\[
\frac{d}{dt} R^2 = \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) = 2 \mathbf{r} \cdot \dot{\mathbf{v}} = 0,
\]
and so $\mathbf{r} \perp \dot{\mathbf{v}}$ (the trivial case $\mathbf{v} = \dot{\theta}$ will not concern us).

The velocity vector is therefore
\[
\dot{\mathbf{v}}(t) = R \frac{d\theta}{dt} \hat{\theta}(t) = R \omega \hat{\theta}(t).
\] (3.2.10)

**Example 2: Relative Motion and Polar Coordinates**

By relative velocity we mean velocity with respect to a specified coordinate system. (The term velocity, alone, is understood to be relative to the observer’s coordinate system.)

![Diagram of a circle with vectors and angular velocities](image)

**a.** A point is observed to have velocity $\dot{\mathbf{v}}_A$ relative to coordinate system $A$. What is its velocity relative to coordinate system $B$, which is displaced from system $A$ by distance $\vec{R}$? ($\vec{R}$ can change in time.)

**b.** Particles $a$ and $b$ move in opposite directions around a circle with the magnitude of the angular velocity $\omega$, as shown. At $t = 0$ they are both at the point $\mathbf{r} = \hat{l} j$, where $l$ is the radius of the circle. Find the velocity of $a$ relative to $b$.

**Solution: (a)** The position vectors are related by
\[
\mathbf{r}_B = \mathbf{r}_A - \vec{R}.
\] (3.3.1)
Then velocities are related by the taking derivatives, (law of addition of velocities)

\[ \vec{v}_B = \vec{v}_A - \vec{V}. \] (3.3.2)

(b) Let’s choose two reference frames; frame B is centered at particle b, and frame A is centered at the center of the circle in the figure below.

Then the relative position vector between the origins of the two frames is given by

\[ \vec{R} = l \hat{r}. \] (3.3.3)

The position vector of particle \(a\) relative to frame \(A\) is given by

\[ \vec{r}_A = l \hat{r}'. \] (3.3.4)

The position vector of particle \(b\) in frame \(B\) can be found by substituting Eqs. (3.3.4) and (3.3.3) into Eq. (3.3.1),

\[ \vec{r}_B = \vec{r}_A - \vec{R} = l \hat{r}' - l \hat{r}. \] (3.3.5)

We can decompose each of the unit vectors \(\hat{r}\) and \(\hat{r}'\) with respect to the Cartesian unit vectors \(\hat{i}\) and \(\hat{j}\) (see figure)

\[ \hat{r} = -\sin \theta \hat{i} + \cos \theta \hat{j} \] (3.3.6)

\[ \hat{r}' = \sin \theta \hat{i} + \cos \theta \hat{j}. \] (3.3.7)

Then Eq. (3.3.5) giving the position vector of particle \(b\) in frame \(B\) becomes

\[ \vec{r}_B = l \hat{r}' - l \hat{r} = l (\sin \theta \hat{i} + \cos \theta \hat{j}) - l (-\sin \theta \hat{i} + \cos \theta \hat{j}) = 2l \sin \theta \hat{i}. \] (3.3.8)

In order to find the velocity vector of particle \(a\) in frame \(B\) (i.e. with respect to particle \(b\)), differentiate Eq. (3.3.8)
\[ \ddot{v}_b = \frac{d}{dt} (2l \sin \theta) \hat{i} = (2l \cos \theta) \frac{d \theta}{dt} \hat{i} = 2 \omega l \cos \theta \hat{i}. \]  

(3.3.9)

### 3.3 Circular Motion: Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component, \( a_{\text{tan}} \equiv a_\theta \), and the radial component, \( a_{\text{rad}} \equiv a_r \). We can write the acceleration vector as

\[ \ddot{a} = a_r \dot{r} + a_\theta \dot{\theta}. \]  

(3.3.10)

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity is changing in magnitude due to the presence of some tangential force. The tangential velocity is \( v_\theta = R \omega \), where \( \omega \) is the angular velocity; if the angular velocity is changing, the velocity is also changing.

Since the radius is constant, the *average tangential acceleration* is just the rate of change of the magnitude of the velocity in a time interval \( \Delta t \),

\[ (a_\theta)_{\text{ave}} = \frac{\Delta v_\theta}{\Delta t} = R \frac{\Delta \omega}{\Delta t}. \]  

(3.3.11)

The instantaneous tangential acceleration involves the same limit argument that we have previously used. Let the time interval \( \Delta t \to 0 \). Then the tangential acceleration is the radius times the derivative of the angular velocity with respect to time,

\[ a_\theta = \lim_{\Delta t \to 0} \frac{\Delta v_\theta}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = R \frac{d\omega}{dt} = R \frac{d^2\theta}{dt^2} \equiv R \alpha. \]  

(3.3.12)

**Definition: Angular Acceleration**

The *angular acceleration* is the rate of change of angular velocity with time (also the second derivative of angle with time) and is denoted by

\[ \alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \]  

(3.3.13)

The SI units of angular acceleration are \([\text{rad} \cdot \text{s}^{-2}]\).

The tangential component of the acceleration is then

\[ a_\theta = R \alpha. \]  

(3.3.14)
**Period and Frequency for Uniform Circular Motion**

If the object is constrained to move in a circle and the total tangential force acting on the object is zero, \( F_{\text{total}}^t = 0 \). By Newton’s Second Law, the tangential acceleration is zero,

\[
a_t = 0.
\]

(3.3.15)

This means that the magnitude of the velocity (the speed) remains constant. This motion is known as uniform circular motion.

Since the speed \( v \) is constant, the amount of time that the object takes to complete one circular orbit of radius \( R \) is also constant. This time interval, \( T \), is called the period. In one period the object travels a distance \( s = vT \) equal to the circumference, \( s = 2\pi R \); thus

\[
s = 2\pi R = vT.
\]

(3.3.16)

The period \( T \) is then given by

\[
T = \frac{2\pi R}{v} = \frac{2\pi R}{\omega} = \frac{2\pi}{|\omega|}.
\]

(3.3.17)

The **frequency** \( f \) is defined to be the reciprocal of the period,

\[
f = \frac{1}{T} = \frac{\omega}{2\pi}.
\]

(3.3.18)

The SI unit of frequency is the inverse second, which is defined as the hertz, \([s^{-1}] = [\text{Hz}]\).

**Radial Acceleration for Uniform Circular Motion**

Of course, not all objects in circular orbits have constant speed. A varying speed will result in a nonzero tangential acceleration \( \alpha \), as described above. A radial acceleration is present due to the change in direction of the velocity vector, and may be calculated by considering the radial component of the change in the velocity vector. Considering only this change, we will assume constant speed, the case of uniform circular motion.

An object traveling in a circular orbit with a constant speed is always accelerating towards the center. Any radial inward acceleration is called **centripetal acceleration**. The magnitude of the velocity is a constant, and the direction of the velocity is always tangent to the circle. However the direction of the velocity is constantly changing because the object is moving in a circle, as can be seen in Figure 3.3.1. Because the velocity changes direction, the object has a nonzero acceleration.
The calculation of the magnitude and direction of the acceleration is very similar to the calculation for the magnitude and direction of the velocity for circular motion, but the change in velocity vector, \( \Delta \vec{v} \), is more complicated to visualize. The change in velocity \( \Delta \vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t) \) is depicted in Figure 3.3.2.

![Figure 3.3.2 Change in velocity vector.](image)

In Figure 3.3.2, the velocity vectors have been given a common point for the tails, so that the change in velocity, \( \Delta \vec{v} \), can be visualized. The length \( |\Delta \vec{v}| \) of the vertical vector can be calculated in exactly the same way as the displacement \( |\Delta \vec{r}| \).

The magnitude of the change in velocity is

\[
|\Delta \vec{v}| = 2v \sin(\Delta \theta / 2).
\] (3.3.19)

We can use the small angle approximation \( \sin(\Delta \theta / 2) \approx \Delta \theta / 2 \) to approximate the magnitude of the change of velocity,

\[
|\Delta \vec{v}| \approx v |\Delta \theta|.
\] (3.3.20)

The magnitude of the radial acceleration is given by

\[
a_r = \lim_{\Delta t \to 0} \frac{|\Delta \vec{v}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\Delta \vec{v}|}{\Delta t} = v \lim_{\Delta \theta \to 0} \frac{|\Delta \theta|}{\Delta t} = v \left| \frac{d \theta}{dt} \right| = v |\omega|.
\] (3.3.21)
The centripetal acceleration can be expressed in several equivalent forms since both the magnitude of the velocity and the angular velocity are related by \( v = R|\omega| \). Thus we have several alternative forms for the magnitude of the centripetal acceleration. The first is that in Equation (3.3.21). The second is in terms of the radius and the angular velocity,

\[
|a_r| = R\omega^2. \tag{3.3.22}
\]

The third form expresses the magnitude of the centripetal acceleration in terms of the speed and radius,

\[
|a_r| = \frac{v^2}{R}. \tag{3.3.23}
\]

Recall that the magnitude of the angular velocity is related to the frequency by \( |\omega| = 2\pi f \), so we have a fourth alternate expression for the magnitude of the centripetal acceleration in terms of the radius and frequency,

\[
|a_r| = 4\pi^2 R f^2. \tag{3.3.24}
\]

A fifth form commonly encountered uses the fact that the frequency and period are related by \( f = 1/T = |\omega|/(2\pi) \). Thus we have the fourth expression for the centripetal acceleration in terms of radius and period,

\[
|a_r| = \frac{4\pi^2 R}{T^2}. \tag{3.3.25}
\]

Other forms, such as \( 4\pi^2 R f^2 / T \) or \( 2\pi R\omega f \), while valid, are uncommon.

Often we decide which expression to use based on information that describes the orbit. A convenient measure might be the orbit’s radius. We may also independently know the period, or the frequency, or the angular velocity, or the speed. If we know one, we can calculate the other three but it is important to understand the meaning of each quantity.

The direction of the acceleration is determined by the same method as the direction of the velocity: in the limit \( \Delta \theta \to 0 \), \( \Delta \vec{v} \perp \vec{v} \), and so the direction of the velocity \( \vec{a} \) at time \( t \) is perpendicular to position vector \( \vec{v}(t) \) and directed inward, in the \( -\hat{r} \)-direction.

So for an object that is undergoing circular motion the acceleration vector has radial and tangential components given by
\[ \mathbf{a}(t) = -R \left( \frac{d\theta}{dt} \right)^2 \mathbf{\hat{r}}(t) + R \frac{d^2\theta}{dt^2} \mathbf{\hat{\theta}}(t) = -R \omega^2 \mathbf{\hat{r}}(t) + R \alpha \mathbf{\hat{\theta}}(t). \]  

(3.3.26)

where \( \omega = \frac{d\theta}{dt} \) is the angular velocity and \( \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \) is the angular acceleration. Keep in mind that as the object moves in a circular, the unit vectors \( \mathbf{\hat{r}}(t) \) and \( \mathbf{\hat{\theta}}(t) \) change direction and hence are not constant in time.

When the motion is uniform circular motion then \( \frac{d\omega}{dt} = 0 \). Therefore the acceleration has only a radial component and the direction is towards the center of the circular orbit

\[ \mathbf{a}(t) = a_r \mathbf{\hat{r}}(t) = -R \omega^2 \mathbf{\hat{r}}(t) = -\frac{v^2}{R} \mathbf{\hat{r}}(t). \]  

(3.3.27)

For an algebraic way of deriving the above results, see Appendix A.

### 3.4 Summary: Circular Motion Kinematics

For an object undergoing circular motion the position vector is given by Eq. (3.2.1)

\[ \mathbf{r}(t) = R \mathbf{\hat{r}}. \]  

(3.4.1)

The arc length for circular motion of radius \( R \) is

\[ s = R \theta. \]  

(3.4.2)

The rate of change of arc length with respect to time is the tangential speed \( v \),

\[ v = \frac{ds}{dt} = R \left| \frac{d\theta}{dt} \right| = R |\omega|, \]  

(3.4.3)

where \( \omega \) is the angular velocity. The velocity vector is given by Eq. (3.2.10)

\[ \mathbf{v}(t) = R \frac{d\theta}{dt} \mathbf{\hat{\theta}}(t) = R \omega \mathbf{\hat{\theta}}(t). \]  

(3.4.4)

The rate of change of the magnitude of the tangential velocity with respect to time is the tangential acceleration

\[ a_t = \frac{dv_t}{dt} = R \frac{d^2\theta}{dt^2} = R \alpha. \]  

(3.4.5)

where \( \alpha \) is the angular acceleration.
The rate of change of the direction of the tangential velocity with respect to time is the centripetal acceleration; this vector is directed towards the center and has magnitude

\[ |a_t| = |v\omega| = \frac{v^2}{R} = R\omega^2. \]  

(3.4.6)

The acceleration vector is given by Eq. (3.3.26)

\[ \ddot{\mathbf{a}}(t) = -R\left(\frac{d\theta}{dt}\right)^2 \mathbf{r}(t) + R\frac{d^2\theta}{dt^2} \mathbf{\hat{r}}(t) = -R\omega^2 \mathbf{r}(t) + R\frac{d\omega}{dt} \mathbf{\hat{\theta}}(t). \]  

(3.4.7)

For uniform circular motion, the acceleration vector is given by Eq. (3.3.27)

\[ \ddot{\mathbf{a}}(t) = -R\omega^2 \mathbf{r}(t). \]  

(3.4.8)

### 3.5 Newton’s Second Law and Circular Motion Dynamics

We have already shown that when an object moves in a circular orbit of radius \( R \) with angular velocity \( \omega \), it is most convenient to choose polar coordinates to describe the position, velocity and acceleration vectors. In particular, the acceleration vector is given by

\[ \ddot{\mathbf{a}} = -R\left(\frac{d\theta}{dt}\right)^2 \mathbf{r} + R\frac{d^2\theta}{dt^2} \mathbf{\hat{r}} = -R\omega^2 \mathbf{r} + R\alpha \mathbf{\hat{r}}. \]  

(3.5.1)

Then Newton’s Second Law

\[ \mathbf{F} = m\ddot{\mathbf{r}}. \]  

(3.5.2)

can be decomposed into radial and tangential components

\[ F_r = -mR\omega^2 \text{ (circular motion).} \]  

(3.5.3)

\[ F_\theta = mR\alpha \text{ (circular motion).} \]  

(3.5.4)

For the special case of uniform circular motion, \( \alpha = 0 \) and so the sum of the tangential components of the force acting on the object is must therefore be zero,

\[ F_\theta = 0 \text{ (uniform circular motion).} \]  

(3.5.5)
3.6 Universal Law of Gravitation and the Circular Orbit of the Moon

An important example of (approximate) circular motion is the orbit of the Moon around the Earth. We can approximately calculate the time \( T \) the Moon takes to complete one circle around the earth (a calculation of great importance to early lunar calendar systems, which became the basis for our current model.) Denote the distance from the moon to the center of the earth by \( R_{c,m} \).

Since the Moon moves in a circle, it is accelerating towards the Earth. The radial acceleration has magnitude

\[
a_r = \frac{4\pi^2 R_{c,m}}{T^2}.
\]  

(3.6.1)

According to Newton’s Second Law, there must be a centripetal force acting on the Moon directed towards the center of the Earth that accounts for this inward acceleration.

Universal Law of Gravitation

Newton’s Universal Law of Gravitation describes the gravitational force between two bodies 1 and 2 with masses \( m_1 \) and \( m_2 \) respectively. This force is a radial force (always pointing along the radial line connecting the masses) and the magnitude is proportional to the inverse square of the distance that separates the bodies. Then the force on body 1 due to the gravitational interaction between the bodies is given by

\[
\vec{F}_{1,2} = -G \frac{m_1 m_2}{r_{1,2}^2} \hat{r}_{1,2},
\]  

(3.6.2)

where \( r_{1,2} \) is the distance between the two bodies and \( \hat{r}_{1,2} \) is the unit vector located at the position of body 1 and pointing from body 2 towards body 1. The Universal Gravitation Constant is \( G = 6.67 \times 10^{-11} \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \). Figure 3.6.1 shows the direction of the forces on bodies 1 and 2 along with the unit vector \( \hat{r}_{1,2} \).

![Figure 3.6.1 Gravitational force of interaction between two bodies](image)
Newton realized that there were still some subtleties involved. First, why should the mass of the Earth act as if it were all placed at the center? Newton showed that for a perfect sphere with uniform mass distribution, all the mass may be assumed to be located at the center. We assume for the present calculation that the Earth and the Moon are perfect spheres with uniform mass distribution.

Second, does this gravitational force between the Earth and the Moon form an action-reaction Third Law pair? When Newton first explained the Moon’s motion in 1666, he had still not formulated the Third Law, which accounted for the long delay in the publication of the *Principia*. The link between the concept of force and the concept of an action-reaction pair of forces was the last piece needed to solve the puzzle of the effect of gravity on planetary orbits. Once Newton realized that the gravitational force between any two bodies forms an action-reaction pair, and satisfies both the Universal Law of Gravitation and his newly formulated Third Law, he was able to solve the oldest and most important physics problem of his time, the motion of the planets.

The test for the Universal Law of Gravitation was performed through experimental observation of the motion of planets, which turned out to be resoundingly successful. For almost 200 years, Newton’s Universal Law was in excellent agreement with observation. A sign of more complicated physics ahead, the first discrepancy only occurred when a slight deviation of the motion of Mercury was experimentally confirmed in 1882. The prediction of this deviation was the first success of Einstein’s Theory of General Relativity (formulated in 1915).

We can apply this Universal Law of Gravitation to calculate the period of the Moon’s orbit around the Earth. The mass of the Moon is $m_1 = 7.36 \times 10^{22}$ kg and the mass of the Earth is $m_2 = 5.98 \times 10^{24}$ kg. The distance from the Earth to the Moon is $R_{e,m} = 3.82 \times 10^8$ m. Newton’s Second Law of motion for the radial direction becomes

$$G \frac{m_1 m_2}{R_{e,m}^2} = m_1 \frac{4\pi^2 R_{e,m}}{T^2}. \quad (3.6.3)$$

We can solve this equation for the period of the orbit,

$$T = \sqrt{\frac{4\pi^2 R_{e,m}^3}{G m_2}}. \quad (3.6.4)$$

Substitute the given values for the radius of the orbit, the mass of the earth, and the universal gravitational constant. The period of the orbit is

$$T = \sqrt{\frac{4\pi^2 (3.82 \times 10^8 \text{ m})^3}{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})}} = 2.35 \times 10^6 \text{ s}. \quad (3.6.5)$$
This period is given in days by

\[ T = \left( 2.35 \times 10^6 \text{s} \right) \left( \frac{1 \text{ day}}{8.64 \times 10^4 \text{s}} \right) = 27.2 \text{ days}. \]  

(3.6.6)

This period is called the sidereal month because it is the time that it takes for the Moon to return to a given position with respect to the stars.

The actual time \( T_1 \) between full moons, called the synodic month (the average period of the Moon’s revolution with respect to the sun and is 29.53 days, it may range between 29.27 days and 29.83 days), is longer than the sidereal month because the Earth is traveling around the Sun. So for the next full moon the Moon must travel a little farther than one full circle around the Earth in order to be on the other side of the Earth from the Sun.

Therefore the time \( T_1 \) between consecutive full moons is approximately \( T_1 = T + \Delta T \) where \( \Delta T = T / 12 = 2.3 \text{ days} \). So \( T_1 = 29.5 \text{ days} \).

**Kepler’s Third Law and Circular Motion**

The first thing that we notice from the above solution is that the period does not depend on the mass of the Moon. We also notice that the square of the period is proportional to the cube of the distance between the Earth and the Moon,

\[ T^2 = \frac{4\pi^2 R_{\text{e,m}}^3}{G m_z}. \]

(3.6.7)

This is an example of Kepler’s Third Law, of which Newton was aware. This confirmation was convincing evidence to Newton that his Universal Law of Gravitation was the correct mathematical description of the gravitational force law, even though he still could not explain what “caused” gravity.
3.7 Worked Examples Circular Motion

Example 1: A geostationary satellite goes around the earth once every 23 hours 56 minutes and 4 seconds, (a sidereal day, shorter than the noon-to-noon solar day of 24 hours) so that its position appears stationary with respect to a ground station. The mass of the earth is \( m_e = 5.98 \times 10^{24} \) kg. The mean radius of the earth is \( R_e = 6.37 \times 10^6 \) m. The universal constant of gravitation is \( G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \). Your goal is to find the radius of the orbit of a geostationary satellite. Describe what motion models this problem. What is the radius of the orbit of a geostationary satellite? Approximately how many earth radii is this distance?

Solution: The satellite’s motion can be modeled as uniform circular motion. The gravitational force between the earth and the satellite keeps the satellite moving in a circle. The acceleration of the satellite is directed towards the center of the circle, that is, along the radially inward direction.

The figure below is close to a scale drawing.

Choose the origin at the center of the earth, and the unit vector \( \hat{r} \) along the radial direction. This choice of coordinates makes sense in this problem since the direction of acceleration is along the radial direction.

Let \( \vec{r} \) be the position vector of the satellite. The magnitude of \( \vec{r} \) (we denote it as \( r_e \)) is the distance of the satellite from the center of the earth, and hence the radius of its circular orbit. Let \( \omega \) be the angular velocity of the satellite, and the period is \( T = \frac{2\pi}{\omega} \). The acceleration is directed inward, with magnitude \( r_e \omega^2 \); in vector form,

\[
\vec{a} = -r_e \omega^2 \hat{r}.
\]  

(3.7.1)
Apply Newton’s Second Law to the satellite for the radial component. The only force in this direction is the gravitational force due to the Earth,

\[ \vec{F}_{\text{grav}} = -m_s \omega^2 r_s \hat{r}. \]  

(3.7.2)

The inward radial force on the satellite is the gravitational attraction of the earth,

\[ -G \frac{m_m c}{r_s^2} \hat{r} = -m_s \omega^2 r_s \hat{r}. \]  

(3.7.3)

Equating the \( \hat{r} \) components,

\[ G \frac{m_m c}{r_s^2} = m_s \omega^2 r_s. \]  

(3.7.4)

Solving for the radius of orbit of the satellite \( r_s \),

\[ r_s = \left( \frac{G m_c}{\omega^2} \right)^{1/3}. \]  

(3.7.5)

The period \( T \) of the satellite’s orbit in seconds is 86164 s and so the angular speed is

\[ \omega = \frac{2\pi}{T} = \frac{2\pi}{86164 \text{ s}} = 7.2921 \times 10^{-5} \text{ s}^{-1}. \]  

(3.7.6)

Using the values of \( \omega, G \) and \( m_c \) in Equation (3.7.5), we determine \( r_s \);

\[ r_s = 4.22 \times 10^7 \text{ m} = 6.62 R_e. \]  

(3.7.7)

**Example 2:** A coin of mass \( m \) (which you may treat as a point object) lies on a turntable, exactly at the rim, a distance \( R \) from the center. The turntable turns at constant angular speed \( \omega \) and the coin rides without slipping. Suppose the coefficient of static friction between the turntable and the coin is given by \( \mu \). Let \( g \) be the gravitational constant. What is the maximum angular speed \( \omega_{\text{max}} \) such that the coin does not slip?

**Solution:** The coin undergoes circular motion at constant speed so it is accelerating inward. The force inward is static friction and at the just slipping point it has reached its
maximum value. We can use Newton’s Second Law to find the maximum angular speed $\omega_{\text{max}}$. We choose a polar coordinate system and the free body force diagram is shown in the figure below.

The contact force is given by

$$\vec{C} = \vec{N} + \vec{f}_s = N\hat{k} - f_s\hat{r}.$$  \hfill (3.7.8)

The gravitational force is given by

$$\vec{F}_{\text{grav}} = -mg\hat{k}.$$  \hfill (3.7.9)

Newton’s Second Law in the radial direction is given by

$$-f_s = -mR\omega^2.$$  \hfill (3.7.10)

Newton’s Second Law, $F_z = ma_z$, in the $z$-direction, noting that the disc is static hence $a_z = 0$, is given by

$$N - mg = 0.$$  \hfill (3.7.11)

Thus the normal force is

$$N = mg.$$  \hfill (3.7.12)

As $\omega$ increases, the static friction increases in magnitude until at $\omega = \omega_{\text{max}}$ and static friction reaches its maximum value (noting Eq. (3.7.12)).

$$(f_s)_{\text{max}} = \mu N = \mu mg.$$  \hfill (3.7.13)

At this value the disc slips. Thus substituting this value for the maximum static friction into Eq. (3.7.10) yields

$$\mu mg = mR\omega_{\text{max}}^2.$$  \hfill (3.7.14)

We can now solve Eq. (3.7.14) for maximum angular speed $\omega_{\text{max}}$ such that the coin does not slip.
\[ \omega_{\text{max}} = \sqrt{\frac{\mu g}{R}}. \]  

(3.7.15)
Example 3 A U-control airplane of mass $M$ is attached by wires of length $L$ and negligible mass to the “pilot” who controls the lift provided by the wing. (The wires control the plane’s elevator.) The plane’s engine keeps it moving at constant speed $v$.

a) Briefly describe how you intend to model the motion of the object. What directions are you choosing for analyzing the components of your forces? Give your reasons.

b) Find the total tension $T$ in the wires when the plane is flown overhead in a circle so that the wires make an angle $\theta$ with the ground. Remember that the wings can provide lift only in the direction perpendicular to their area, that is, in a direction perpendicular to the wires. Think carefully before selecting the directions of the axes of your coordinate system.

c) The plane will go out of control and crash if the tension is not maintained. Given a particular speed $v$ of the plane, is there some angle $\theta_{crit}$ which you would advise the pilot not to exceed? If possible, exhibit a speed $v_{safe}$ at which the plane would be safe at any angle.

Solution:

a) The “object” (the airplane) is moving in a horizontal circle. The radius of the circle and the hence the period, for a given speed, will depend on the angle that the wires make with respect to the horizontal. Anticipating parts b) and c), the radius of the circle will be $r = L \cos \theta$ and hence the period of the circular motion will be $2\pi r / v = 2\pi L \cos \theta / v$.

One choice of coordinates is that shown in the free body diagram on the left below, with $\mathbf{k}$ directed vertically upward and $\mathbf{r}$ directed radially outward. Two advantages to this choice are that the acceleration and the gravitational force are along the coordinate directions, and that this is the “standard,” perhaps used in previous problems.

Another choice would be that shown on the right below. The advantage to this system is that the unknown forces, including the tension to be determined in part b), are along the coordinate axes, and need not be decomposed. However, the weight $m\mathbf{g}$ and the net force must be decomposed along the given axes.
b) We’ll do this using both of the coordinate systems, starting with the “tilted” system on the right below. Note that the \( \mathbf{i} \)- and \( \mathbf{j} \)-directions are not the horizontal and vertical directions (except for the case \( \theta = 0 \)).

Applying Newton’s Second Law, with \( a = |a_r| \) as the magnitude of the centripetal acceleration, for the two coordinates, we have

\[
\begin{align*}
\mathbf{i} &: \quad T + mg \sin \theta = ma \cos \theta \\
\mathbf{j} &: \quad N - mg \cos \theta = ma \sin \theta.
\end{align*}
\] (3.7.16)

Now we go back to the question asked: We want to find the tension but not the normal force. We know that \( a = v^2 / r = v^2 / L \cos \theta \), so the first expression in (3.7.16) leads almost immediately to

\[
T = m \left( \frac{v^2}{L} - g \sin \theta \right),
\] (3.7.17)

a very tidy result.

If we used the coordinate system on the left above, Newton’s Second Law becomes

\[
\begin{align*}
\mathbf{k} &: \quad N \cos \theta - T \sin \theta = mg \\
\mathbf{r} &: \quad N \sin \theta + T \cos \theta = ma.
\end{align*}
\] (3.7.18)

The expressions in (3.7.18) are two equations in the two unknowns \( N \) and \( T \) and may be solved by standard methods. For our immediate purpose, however, eliminate \( N \) by multiplying the first by \( \sin \theta \) and the second by \( \cos \theta \) and then subtract the first from the second, leading to Equation (3.7.17).

Decide for yourself which coordinate system is “better” or “easier,” but keep in mind that either will work.
c) Either way, we are now set to find the critical angle $\theta_{\text{crit}}$ at which the tension as found in Equation (3.7.17) would go to zero,

$$\theta_{\text{crit}} = \sin^{-1}\left(\frac{v^2}{gL}\right). \quad (3.7.19)$$

We see that if $v^2 > gL$, the critical angle is never reached, and all angles would be safe, so we have

$$v_{\text{safe}} = \sqrt{gL}. \quad (3.7.20)$$

Notice that if $\theta = 0$ the rope lies in the plane of the airplane’s circular orbit, $\sin \theta = 0$, and the tension $T = mv^2 / L > 0$ for all velocities (assuming the normal force $N$, acting vertically upward, is sufficient to keep the plane flying). The other extreme value occurs when $\theta \to \pi / 2$. This corresponds to the radius of the orbit $r \to 0$, $\cos \theta \to 1$ and the tension $T \to \left(\frac{mv^2}{L}\right) - mg$. In order for the tension to stay positive, $v > \sqrt{gL}$ (which is consistent with $\theta_{\text{crit}} \to \pi / 2$).

**Extra:** This problem did not ask for the normal force $N$ (often called the “lift”), but it’s not hard to find

$$N = mg\left(\frac{v^2}{L}\tan \theta + \cos \theta\right). \quad (3.7.21)$$

From this we see that as $\theta \to \pi / 2$, $N \to \infty$ and $r \to 0$, an unphysical result.