Chapter 3 Kinematics

In the first place, what do we mean by time and space? It turns out that these deep philosophical questions have to be analyzed very carefully in physics, and this is not easy to do. The theory of relativity shows that our ideas of space and time are not as simple as one might imagine at first sight. However, for our present purposes, for the accuracy that we need at first, we need not be very careful about defining things precisely. Perhaps you say, “That’s a terrible thing—I learned that in science we have to define everything precisely.” We cannot define anything precisely! If we attempt to, we get into that paralysis of thought that comes to philosophers, who sit opposite each other, one saying to the other, “You don’t know what you are talking about!” The second one says. “What do you mean by know? What do you mean by talking? What do you mean by you?”, and so on. In order to be able to talk constructively, we just have to agree that we are talking roughly about the same thing. You know as much about time as you need for the present, but remember that there are some subtleties that have to be discussed; we shall discuss them later.

Richard Feynman, The Feynman Lectures on Physics

Part A: One-Dimensional Motion

Introduction

Kinematics is the mathematical description of motion. The term is derived from the Greek word kinema, meaning movement. In order to quantify motion, a mathematical coordinate system, called a reference frame, is used to describe space and time. Once a reference frame has been chosen, we can introduce the physical concepts of position, velocity and acceleration in a mathematically precise manner. Figure 3.1 shows a Cartesian coordinate system in one dimension with unit vector \( \hat{i} \) pointing in the direction of increasing \( x \)-coordinate.

![Figure 3.1 A one-dimensional Cartesian coordinate system.](image)

3.1 Position, Time Interval, Displacement

Position

Consider an object moving in one dimension. We denote the position coordinate of the center of mass of the object with respect to the choice of origin by \( x(t) \). The position coordinate is a function of time and can be positive, zero, or negative, depending on the location of the object. The position has both direction and magnitude, and hence is a vector (Figure 3.2),

\[
\mathbf{x}(t) = x(t) \hat{\mathbf{i}}. \tag{3.1.1}
\]

We denote the position coordinate of the center of the mass at \( t = 0 \) by the symbol \( x_0 \equiv x(t = 0) \). The SI unit for position is the meter [m] (see Section 1.3).

![Figure 3.2](image)

**Figure 3.2** The position vector, with reference to a chosen origin.

Time Interval

Consider a closed interval of time \([t_1, t_2]\). We characterize this time interval by the difference in endpoints of the interval such that

\[
\Delta t = t_2 - t_1. \tag{3.1.2}
\]

The SI units for time intervals are seconds [s].

**Definition: Displacement**

The change in position coordinate of the mass between the times \( t_1 \) and \( t_2 \) is

\[
\Delta \mathbf{x} = (x(t_2) - x(t_1)) \hat{\mathbf{i}} = \Delta x(t) \hat{\mathbf{i}}. \tag{3.1.3}
\]

This is called the displacement between the times \( t_1 \) and \( t_2 \) (Figure 3.3). Displacement is a vector quantity.
3.2 Velocity

When describing the motion of objects, words like “speed” and “velocity” are used in common language; however when introducing a mathematical description of motion, we need to define these terms precisely. Our procedure will be to define average quantities for finite intervals of time and then examine what happens in the limit as the time interval becomes infinitesimally small. This will lead us to the mathematical concept that velocity at an instant in time is the derivative of the position with respect to time.

**Definition: Average Velocity**

The component of the average velocity, $v_x$, for a time interval $\Delta t$ is defined to be the displacement $\Delta x$ divided by the time interval $\Delta t$,

$$\bar{v}_x \equiv \frac{\Delta x}{\Delta t}. \quad (3.2.1)$$

The average velocity vector is then

$$\bar{\mathbf{v}}(t) \equiv \frac{\Delta x}{\Delta t} \hat{i} = \bar{v}_x(t) \hat{i}. \quad (3.2.2)$$

The SI units for average velocity are meters per second $[\text{m/s}]$.

**Instantaneous Velocity**

Consider a body moving in one direction. We denote the position coordinate of the body by $x(t)$, with initial position $x_0$ at time $t = 0$. Consider the time interval $[t, t + \Delta t]$. The average velocity for the interval $\Delta t$ is the slope of the line connecting the points $(t, x(t))$ and $(t, x(t + \Delta t))$. The slope, the rise over the run, is the change in position over the change in time, and is given by
\[ \frac{v_x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \]  \hspace{1cm} (3.2.3)

Let’s see what happens to the average velocity as we shrink the size of the time interval. The slope of the line connecting the points \((t, x(t))\) and \((t, x(t + \Delta t))\) approaches the slope of the tangent line to the curve \(x(t)\) at the time \(t\) (Figure 3.4).

**Figure 3.4** Graph of position vs. time showing the tangent line at time \(t\).

In order to define the limiting value for the slope at any time, we choose a time interval \([t, t + \Delta t]\). For each value of \(\Delta t\), we calculate the average velocity. As \(\Delta t \to 0\), we generate a sequence of average velocities. The limiting value of this sequence is defined to be the \(x\)-component of the instantaneous velocity at the time \(t\).

**Definition: Instantaneous Velocity**

The \(x\)-component of instantaneous velocity at time \(t\) is given by the slope of the tangent line to the curve of position vs. time curve at time \(t\):

\[ v_x(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \equiv \frac{dx}{dt}. \]  \hspace{1cm} (3.2.4)

The instantaneous velocity vector is then

\[ \vec{v}(t) = v_x(t) \hat{i}. \]  \hspace{1cm} (3.2.5)

**Example 1: Determining Velocity from Position**

Consider an object that is moving along the \(x\)-coordinate axis represented by the equation
where \( x_0 \) is the initial position of the object at \( t = 0 \).

We can explicitly calculate the \( x \)-component of instantaneous velocity from Equation (3.2.4) by first calculating the displacement in the \( x \)-direction, \( \Delta x = x(t + \Delta t) - x(t) \). We need to calculate the position at time \( t + \Delta t \),

\[
x(t + \Delta t) = x_0 + \frac{1}{2} b(t + \Delta t)^2 = x_0 + \frac{1}{2} b \left( t^2 + 2t\Delta t + \Delta t^2 \right).
\]  

Then the instantaneous velocity is

\[
v_x(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left( x_0 + \frac{1}{2} b \left( t^2 + 2t\Delta t + \Delta t^2 \right) \right) - \left( x_0 + \frac{1}{2} b t^2 \right)}{\Delta t}.
\]  

This expression reduces to

\[
v_x(t) = \lim_{\Delta t \to 0} \left( bt + \frac{1}{2} b \Delta t \right).
\]  

The first term is independent of the interval \( \Delta t \) and the second term vanishes because the limit as \( \Delta t \to 0 \) of \( \Delta t \) is zero. Thus the instantaneous velocity at time \( t \) is

\[
v_x(t) = bt.
\]  

In Figure 3.5 we graph the instantaneous velocity, \( v_x(t) \), as a function of time \( t \).

![Figure 3.5 A graph of instantaneous velocity as a function of time.](image)
3.3 Acceleration

We shall apply the same physical and mathematical procedure for defining acceleration, the rate of change of velocity. We first consider how the instantaneous velocity changes over an interval of time and then take the limit as the time interval approaches zero.

**Average Acceleration**

Acceleration is the quantity that measures a change in velocity over a particular time interval. Suppose during a time interval \( \Delta t \) a body undergoes a change in velocity

\[
\Delta \vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t).
\]  

(3.3.1)

The change in the \( x \)-component of the velocity, \( \Delta v_x \), for the time interval \([t, t + \Delta t]\) is then

\[
\Delta v_x = v_x(t + \Delta t) - v_x(t).
\]  

(3.3.2)

**Definition: Average Acceleration**

The \( x \)-component of the average acceleration for the time interval \( \Delta t \) is defined to be

\[
\bar{a}_x = a_x \hat{i} \equiv \frac{\Delta v_x}{\Delta t} \hat{i} = \frac{(v_x(t + \Delta t) - v_x(t))}{\Delta t} \hat{i} = \frac{\Delta v_x}{\Delta t} \hat{i}.
\]  

(3.3.3)

The SI units for average acceleration are meters per second squared, \([\text{m} \cdot \text{s}^{-2}]\).

**Instantaneous Acceleration**

On a graph of the \( x \)-component of velocity vs. time, the average acceleration for a time interval \( \Delta t \) is the slope of the straight line connecting the two points \((t, v_x(t))\) and \((t + \Delta t, v_x(t + \Delta t))\). In order to define the \( x \)-component of the instantaneous acceleration at time \( t \), we employ the same limiting argument as we did when we defined the instantaneous velocity in terms of the slope of the tangent line.

**Definition: Instantaneous Acceleration.**

The \( x \)-component of the instantaneous acceleration at time \( t \) is the limit of the slope of the tangent line at time \( t \) of the graph of the \( x \)-component of the velocity as a function of time,
The instantaneous acceleration vector is then

\[ \ddot{a}(t) = a_x(t) \hat{i}. \] (3.3.5)

In Figure 3.6 we illustrate this geometrical construction.

**Figure 3.6** Graph of velocity vs. time showing the tangent line at time \( t \).

Since velocity is the derivative of position with respect to time, the \( x \)-component of the acceleration is the second derivative of the position function,

\[ a_x = \frac{d^2 x}{dt^2}. \] (3.3.6)

**Example 2: Determining Acceleration from Velocity**

Let’s continue Example 1, in which the position function for the body is given by \( x = x_0 + (1/2)bt^2 \), and the \( x \)-component of the velocity is \( v_x = bt \). The \( x \)-component of the instantaneous acceleration at time \( t \) is the limit of the slope of the tangent line at time \( t \) of the graph of the \( x \)-component of the velocity as a function of time (Figure 3.5)

\[ a_x = \frac{dv_x}{dt} = \lim_{\Delta t \to 0} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{b(t + \Delta t) - bt}{\Delta t} = b. \] (3.3.7)

Note that in Equation (3.3.7), the ratio \( \Delta v / \Delta t \) is independent of \( \Delta t \), consistent with the constant slope of the graph in Figure 3.4.
3.4 Constant Acceleration

Let’s consider a body undergoing constant acceleration for a time interval $\Delta t = [0, t]$. When the acceleration $a_x$ is a constant, the average acceleration is equal to the instantaneous acceleration. Denote the $x$-component of the velocity at time $t = 0$ by $v_{x,0} \equiv v_x(t = 0)$. Therefore the $x$-component of the acceleration is given by

$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{v_x(t) - v_{x,0}}{t}.$$  \hspace{1cm} (3.4.1)

Thus the velocity as a function of time is given by

$$v_x(t) = v_{x,0} + a_x t. \hspace{1cm} (3.4.2)$$

When the acceleration is constant, the velocity is a linear function of time.

**Velocity: Area Under the Acceleration vs. Time Graph**

In Figure 3.7, the $x$-component of the acceleration is graphed as a function of time.

![Graph of the $x$-component of the acceleration for $a_x$ constant as a function of time.](Figure 3.7)

The area under the acceleration vs. time graph, for the time interval $\Delta t = t - 0 = t$, is

$$\text{Area}(a_x, t) \equiv a_x t.$$  \hspace{1cm} (3.4.3)

Using the definition of average acceleration given above,

$$\text{Area}(a_x, t) \equiv a_x t = \Delta v_x = v_x(t) - v_{x,0}.$$  \hspace{1cm} (3.4.4)
Displacement: Area Under the Velocity vs. Time Graph

In Figure 3.8, we graph the $x$-component of the velocity vs. time curve.

![Graph of velocity as a function of time for $a_x$ constant.]

The region under the velocity vs. time curve is a trapezoid, formed from a rectangle and a triangle and the area of the trapezoid is given by

$$\text{Area}(v_x, t) = v_{x, 0} t + \frac{1}{2} (v_x(t) - v_{x, 0}) t . \quad (3.4.5)$$

Substituting for the velocity (Equation (3.4.2)) yields

$$\text{Area}(v_x, t) = v_{x, 0} t + \frac{1}{2} (v_{x, 0} + a_x t - v_{x, 0}) t = v_{x, 0} t + \frac{1}{2} a_x t^2 . \quad (3.4.6)$$

![The average velocity over a time interval.]

We can then determine the average velocity by adding the initial and final velocities and dividing by a factor of two (see Figure 3.9).

$$\overline{v_x} = \frac{1}{2} (v_x(t) + v_{x, 0}) . \quad (3.4.7)$$
The above method for determining the average velocity differs from the definition of average velocity in Equation (3.2.1). When the acceleration is constant over a time interval, the two methods will give identical results. Substitute into Equation (3.4.7) the $x$-component of the velocity, Equation (3.4.2), to yield

$$
\overrightarrow{v}_x = \frac{1}{2} \left( v_x(t) + v_{x,0} \right) = \frac{1}{2} \left( (v_{x,0} + a_x t) + v_{x,0} \right) = v_{x,0} + \frac{1}{2} a_x t. \tag{3.4.8}
$$

Recall Equation (3.2.1); the average velocity is the displacement divided by the time interval (note we are now using the definition of average velocity that always holds, for non-constant as well as constant acceleration). The displacement is equal to

$$
\Delta x \equiv x(t) - x_0 = \overrightarrow{v}_x \cdot t. \tag{3.4.9}
$$

Substituting Equation (3.4.8) into Equation (3.4.9) shows that displacement is given by

$$
\Delta x \equiv x(t) - x_0 = v_{x,0} t + \frac{1}{2} a_x t^2. \tag{3.4.10}
$$

Now compare Equation (3.4.10) to Equation (3.4.6) to conclude that the displacement is equal to the area under the graph of the $x$-component of the velocity vs. time,

$$
\Delta x \equiv x(t) - x_0 = v_{x,0} t + \frac{1}{2} a_x t^2 = \text{Area}(v_x, t), \tag{3.4.11}
$$

and so we can solve Equation (3.4.11) for the position as a function of time,

$$
x(t) = x_0 + v_{x,0} t + \frac{1}{2} a_x t^2. \tag{3.4.12}
$$

Figure 3.10 shows a graph of this equation. Notice that at $t = 0$ the slope may be in general non-zero, corresponding to the initial velocity component $v_{x,0}$.  

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.10.png}
\caption{Graph showing the position $x(t)$ as a function of time $t$.}
\end{figure}
3.5 Integration and Kinematics

Change of Velocity as the Integral of Non-constant Acceleration

When the acceleration is a non-constant function, we would like to know how the \( x \)-component of the velocity changes for a time interval \( \Delta t = [0, t] \). Since the acceleration is non-constant we cannot simply multiply the acceleration by the time interval. We shall calculate the change in the \( x \)-component of the velocity for a small time interval \( \Delta t_i = [t_i, t_{i+1}] \) and sum over these results. We then take the limit as the time intervals become very small and the summation becomes an integral of the \( x \)-component of the acceleration.

For a time interval \( \Delta t = [0, t] \), we divide the interval up into \( N \) small intervals \( \Delta t_i = [t_i, t_{i+1}] \), where the index \( i = 1, 2, ..., N \), and \( t_i \equiv 0, t_N \equiv t \). Over the interval \( \Delta t_i \), we can approximate the acceleration as a constant, \( \overline{a_x(t_i)} \). Then the change in the \( x \)-component of the velocity is the area under the acceleration vs. time curve,

\[
\Delta v_{x,i} \equiv v_x(t_{i+1}) - v_x(t_i) = \overline{a_x(t_i)} \Delta t_i + E_i
\]  (3.5.1)

where \( E_i \) is the error term (see Figure 3.11a). Then the sum of the changes in the \( x \)-component of the velocity is

\[
\sum_{i=1}^{i=N} \Delta v_{x,i} = (v_x(t_2) - v_x(t_1 = 0)) + (v_x(t_3) - v_x(t_2)) + \cdots + (v_x(t_N - t) - v_x(t_{N-1})).
\]  (3.5.2)

In this summation pairs of terms of the form \( (v_x(t_2) - v_x(t_2)) = 0 \) sum to zero, and the overall sum becomes

\[
v_x(t) - v_x(0) = \sum_{i=1}^{i=N} \Delta v_{x,i}.
\]  (3.5.3)

Substituting Equation (3.5.1) into Equation (3.5.3),

\[
v_x(t) - v_x(0) = \sum_{i=1}^{i=N} \Delta v_{x,i} = \sum_{i=1}^{i=N} \overline{a_x(t_i)} \Delta t_i + \sum_{i=1}^{i=N} E_i.
\]  (3.5.4)

We now approximate the area under the graph in Figure 3.11a by summing up all the rectangular area terms,
\[ \text{Area}_N(a_x, t) = \sum_{i=1}^{\frac{t}{\Delta t}} a_x(t_i) \Delta t_i . \] (3.5.5)

**Figures 3.11a and 3.11b** Approximating the area under the graph of the \( x \)-component of the acceleration vs. time

Suppose we make a finer subdivision of the time interval \( \Delta t = [0, t] \) by increasing \( N \), as shown in Figure 3.11b. The error in the approximation of the area decreases. We now take the limit as \( N \) approaches infinity and the size of each interval \( \Delta t_i \) approaches zero. For each value of \( N \), the summation in Equation (3.5.5) gives a value for \( \text{Area}_N(a_x, t) \), and we generate a sequence of values

\[ \{\text{Area}_1(a_x, t), \text{Area}_2(a_x, t), ..., \text{Area}_N(a_x, t)\} . \] (3.5.6)

The limit of this sequence is the area, \( \text{Area}(a_x, t) \), under the graph of the \( x \)-component of the acceleration vs. time. When taking the limit, the error term vanishes in Equation (3.5.4),

\[ \lim_{N \to \infty} \sum_{i=1}^{\frac{t}{\Delta t}} E_i = 0 . \] (3.5.7)

Therefore in the limit as \( N \) approaches infinity, Equation (3.5.4) becomes

\[ v_x(t) - v_x(0) = \lim_{N \to \infty} \sum_{i=1}^{\frac{t}{\Delta t}} a_x(t_i) \Delta t_i + \lim_{N \to \infty} \sum_{i=1}^{\frac{t}{\Delta t}} E_i = \lim_{N \to \infty} \sum_{i=1}^{\frac{t}{\Delta t}} a_x(t_i) \Delta t_i = \text{Area}(a_x, t) , \] (3.5.8)
and thus the change in the $x$-component of the velocity is equal to the area under the graph of $x$-component of the acceleration vs. time.

**Definition: Integral of acceleration**

The *integral of the $x$-component of the acceleration* for the interval $[0, t]$ is defined to be the limit of the sequence of areas, $\text{Area}_N(a_x, t)$, and is denoted by

$$\int_{t'=0}^{t'} a_x(t') \, dt' \equiv \lim_{\Delta t \to 0} \sum_{i=1}^{i=N} a_x(t_i) \Delta t_i = \text{Area}(a_x, t). \quad (3.5.9)$$

Equation (3.5.8) shows that the change in the $x$-component of the velocity is the integral of the $x$-component of the acceleration with respect to time.

$$v_x(t) - v_x(0) = \int_{t'=0}^{t'} a_x(t') \, dt'. \quad (3.5.10)$$

Using integration techniques, we can in principle find the expressions for the velocity as a function of time for any acceleration.

**Integral of Velocity**

We can repeat the same argument for approximating the area $\text{Area}(v_x, t)$ under the graph of the $x$-component of the velocity vs. time by subdividing the time interval into $N$ intervals and approximating the area by

$$\text{Area}_N(a_x, t) = \sum_{i=1}^{i=N} v_x(t_i) \Delta t_i. \quad (3.5.11)$$

The displacement for a time interval $\Delta t = [0, t]$ is limit of the sequence of sums $\text{Area}_N(a_x, t)$,

$$\Delta x = x(t) - x(0) = \lim_{N \to \infty} \sum_{i=1}^{i=N} v_x(t_i) \Delta t_i. \quad (3.5.12)$$

This approximation is shown in Figure 3.12.
**Figure 3.12** Approximating the area under the graph of the $x$-component of the velocity vs. time.

**Definition: Integral of Velocity**

The integral of the $x$-component of the velocity for the interval $[0, t]$ is the limit of the sequence of areas, $\text{Area}_{x}(a_{x}, t)$, and is denoted by

$$\int_{t=0}^{t'=\Delta t} v_{x}(t') \, dt' \equiv \lim_{\Delta t \to 0} \sum_{i=1}^{i=N} v_{x}(t_{i}) \, \Delta t_{i} = \text{Area}(v_{x}, t). \quad (3.5.13)$$

The displacement is then the integral of the $x$-component of the velocity with respect to time,

$$\Delta x = x(t) - x(0) = \int_{t=0}^{t'=\Delta t} v_{x}(t') \, dt'. \quad (3.5.14)$$

Using integration techniques, we can in principle find the expressions for the position as a function of time for any acceleration.

**Example:**

Let’s consider a case in which the acceleration, $a_{x}(t)$, is not constant in time,

$$a_{x}(t) = b_{0} + b_{1} \, t + b_{2} \, t^{2} \quad (3.5.15)$$

The graph of the $x$-component of the acceleration vs. time is shown in Figure 3.13.
Figure 3.13 A non-constant acceleration vs. time graph.

Let’s find the change in the $x$-component of the velocity as a function of time. Denote the initial velocity at $t = 0$ by $v_{x,0} \equiv v_x(t = 0)$. Then,

$$v_x(t) - v_{x,0} = \int_{t'=0}^{t'} a_x(t') \, dt' = \int_{t'=0}^{t'} (b_0 + b_1 t' + b_2 t'^2) \, dt' = b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \quad (3.5.16)$$

The $x$-component of the velocity as a function in time is then

$$v_x(t) = v_{x,0} + b_0 t + \frac{b_1 t^2}{2} + \frac{b_2 t^3}{3}. \quad (3.5.17)$$

Denote the initial position by $x_0 \equiv x(t = 0)$. The displacement as a function of time is the integral

$$x(t) - x_0 = \int_{t'=0}^{t'} v_x(t') \, dt'. \quad (3.5.18)$$

Use Equation (3.5.17) for the $x$-component of the velocity in Equation (3.5.18) to find

$$x(t) - x_0 = \int_{t'=0}^{t'} \left( v_{x,0} + b_0 t' + \frac{b_1 t'^2}{2} + \frac{b_2 t'^3}{3} \right) \, dt' = v_{x,0} t + \frac{b_0 t^2}{2} + \frac{b_1 t^3}{6} + \frac{b_2 t^4}{12}. \quad (3.5.19)$$

Finally the position is then
3.6 Free Fall

An important example of one-dimensional motion (for both scientific and historical reasons) is an object undergoing free fall. Suppose you are holding a stone and throw it straight up in the air. For simplicity, we’ll neglect all the effects of air resistance. The stone will rise and fall along a line, and so the stone is moving in one dimension.

Galileo Galilei was the first to definitively state that all objects fall towards the earth with a constant acceleration, later measured to be of magnitude $g = 9.8 \, \text{m/s}^2$ to two significant figures (see Section 4.3). The symbol $g$ will always denote the magnitude of the acceleration at the surface of the earth. (We will later see that Newton’s Universal Law of Gravitation requires some modification of Galileo’s statement, but near the earth’s surface his statement holds.) Let’s choose a coordinate system with the origin located at the ground, and the $y$-axis perpendicular to the ground with the $y$-coordinate increasing in the upward direction. With our choice of coordinate system, the acceleration is constant and negative,

$$a_y(t) = -g = -9.8 \, \text{m/s}^2. \quad (3.6.1)$$

*When we ignore the effects of air resistance, the acceleration of any object in free fall near the surface of the earth is downward, constant and equal to $9.8 \, \text{m/s}^2$. Of course, if more precise numerical results are desired, a more precise value of $g$ must be used (see Section 4.3).*

**Equations of Motions**

We have already determined the position equation (Equation (3.4.12)) and velocity equation (Equation (3.4.2)) for an object undergoing constant acceleration. With a simple change of variables from $x \rightarrow y$, the two equations of motion for a freely falling object are

$$y(t) = y_0 + v_{y,0} t - \frac{1}{2} g t^2 \quad (3.6.2)$$

and

$$v_y(t) = v_{y,0} - g t, \quad (3.6.3)$$
where $y_0$ is the initial position from which the stone was released at $t = 0$, and $v_{y,0}$ is the initial $y$-component of velocity that the stone acquired at $t = 0$ from the act of throwing.

**Part B: Two-Dimensional Motion**

### 3.7 Introduction to the Vector Description of Motion in Two and Three Dimensions

So far we have introduced the concepts of kinematics to describe motion in one dimension; however we live in a multidimensional universe. In order to explore and describe motion in this universe, we begin by looking at examples of two-dimensional motion, of which there are many; planets orbiting a star in elliptical orbits or a projectile moving under the action of uniform gravitation are two common examples.

We will now extend our definitions of position, velocity, and acceleration for an object that moves in two dimensions (in a plane) by treating each direction independently, which we can do with vector quantities by resolving each of these quantities into components. For example, our definition of velocity as the derivative of position holds for each component separately. In Cartesian coordinates, in which the directions of the unit vectors do not change from place to place, the position vector $\mathbf{r}(t)$ with respect to some choice of origin for the object at time $t$ is given by

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}. \quad (3.7.1)$$

The velocity vector $\mathbf{v}(t)$ at time $t$ is the derivative of the position vector,

$$\mathbf{v}(t) = \frac{dx(t)}{dt} \mathbf{i} + \frac{dy(t)}{dt} \mathbf{j} \equiv v_x(t) \mathbf{i} + v_y(t) \mathbf{j}, \quad (3.7.2)$$

where $v_x(t) \equiv dx(t)/dt$ and $v_y(t) \equiv dy(t)/dt$ denote the $x$- and $y$-components of the velocity respectively.

The acceleration vector $\mathbf{a}(t)$ is defined in a similar fashion as the derivative of the velocity vector,

$$\mathbf{a}(t) = \frac{dv_x(t)}{dt} \mathbf{i} + \frac{dv_y(t)}{dt} \mathbf{j} \equiv a_x(t) \mathbf{i} + a_y(t) \mathbf{j}, \quad (3.7.3)$$

where $a_x(t) \equiv dv_x(t)/dt$ and $a_y(t) \equiv dv_y(t)/dt$ denote the $x$- and $y$-components of the acceleration.
3.8 Projectile Motion

A special case of two-dimensional motion occurs when the vertical component of the acceleration is constant and the horizontal component is zero. Then the complete set of equations for position and velocity for each independent direction of motion are given by

\[
\begin{align*}
\mathbf{r}(t) &= x(t) \hat{i} + y(t) \hat{j} = (x_0 + v_{x,0} t) \hat{i} + \left( y_0 + v_{y,0} t + \frac{1}{2} a_y t^2 \right) \hat{j}, \\
\mathbf{v}(t) &= v_x(t) \hat{i} + v_y(t) \hat{j} = v_{x,0} \hat{i} + \left( v_{y,0} + a_y t \right) \hat{j}, \\
\mathbf{a}(t) &= a_x(t) \hat{i} + a_y(t) \hat{j} = a_y \hat{j}.
\end{align*}
\]

(3.8.1) \hspace{1cm} (3.8.2) \hspace{1cm} (3.8.3)

Consider the motion of a body that is released with an initial velocity \( \mathbf{v}_0 \) at a height \( h \) above the ground. Two paths are shown in Figure 3.14.

![Figure 3.14](image)

**Figure 3.14** Actual orbit and parabolic orbit of a projectile

The dotted path represents a parabolic trajectory and the solid path represents the actual orbit. The difference between the paths is due to air resistance. There are other factors that can influence the path of motion; a rotating body or a special shape can alter the flow of air around the body, which may induce a curved motion or lift like the flight of a baseball or golf ball. We shall begin our analysis by neglecting all influences on the body except for the influence of gravity.

We shall choose coordinates with our \( y \)-axis in the vertical direction with \( \hat{j} \) directed upwards and our \( x \)-axis in the horizontal direction with \( \hat{i} \) directed in the direction that the body is moving horizontally. We choose our origin to be the place where the body is released at time \( t = 0 \). Figure 3.15 shows our coordinate system with the position of the body at time \( t \) and the coordinate functions \( x(t) \) and \( y(t) \).
The coordinate function $y(t)$ represents the distance from the body to the origin along the $y$-axis at time $t$, and the coordinate function $x(t)$ represents the distance from the body to the origin along the $x$-axis at time $t$.

The $y$-component of the acceleration,

$$a_y = -g,$$  

is a constant and is independent of the mass of the body. Notice that $a_y < 0$; this is because we chose our positive $y$-direction to point upwards.

Since we are ignoring the effects of any horizontal forces, the acceleration in the horizontal direction is zero,

$$a_x = 0;$$  

therefore the $x$-component of the velocity remains unchanged throughout the flight.

**Kinematic Equations of Motion**

The kinematic equations of motion for the position and velocity components of the object are

$$x(t) = x_0 + v_{x,0} t,$$  

$$v_x(t) = v_{x,0},$$  

$$y(t) = y_0 + v_{y,0} t - \frac{1}{2} g t^2,$$
\[ v_y(t) = v_{y,0} - gt. \] (3.8.9)

**Initial Conditions**

In these equations, the initial velocity vector is

\[ \vec{v}_0(t) = v_{x,0} \hat{i} + v_{y,0} \hat{j}. \] (3.8.10)

Often the description of the flight of a projectile includes the statement, “a body is projected with an initial speed \( v_0 \) at an angle \( \theta_0 \) with respect to the horizontal.” The vector decomposition diagram for the initial velocity is shown in Figure 3.16. The components of the initial velocity are given by

\[ v_{x,0} = v_0 \cos \theta_0, \] (3.8.11)

\[ v_{y,0} = v_0 \sin \theta_0. \] (3.8.12)

![Figure 3.16 A vector decomposition of the initial velocity](image)

Since the initial speed is the magnitude of the initial velocity, we have

\[ v_0 = \left( v_{x,0}^2 + v_{y,0}^2 \right)^{1/2}. \] (3.8.13)

The angle \( \theta_0 \) is related to the components of the initial velocity by

\[ \theta_0 = \tan^{-1} \left( \frac{v_{y,0}}{v_{x,0}} \right). \] (3.8.14)

The initial position vector appears with components

\[ \vec{r}_0 = x_0 \hat{i} + y_0 \hat{j}. \] (3.8.15)
Note that the trajectory in Figure 3.16 has \( x_0 = y_0 = 0 \), but this will not always be the case, as in the analysis below.

**Orbit equation**

So far our description of the motion has emphasized the independence of the spatial dimensions, treating all of the kinematic quantities as functions of time. We shall now eliminate time from our equation and find the *orbit equation* of the body. We begin with Equation (3.8.6) for the \( x \)-component of the position,

\[
x(t) = x_0 + v_{x,0} t
\]  

(3.8.16)

and solve Equation (3.8.16) for time \( t \) as a function of \( x(t) \),

\[
t = \frac{x(t) - x_0}{v_{x,0}}.
\]  

(3.8.17)

The vertical position of the body is given by Equation (3.8.8),

\[
y(t) = y_0 + v_{y,0} t - \frac{1}{2} g t^2.
\]  

(3.8.18)

We then substitute the above expression, Equation (3.8.17) for time \( t \) into our equation for the \( y \)-component of the position yielding

\[
y(t) = y_0 + v_{y,0} \left( \frac{x(t) - x_0}{v_{x,0}} \right) - \frac{1}{2} g \left( \frac{x(t) - x_0}{v_{x,0}} \right)^2.
\]  

(3.8.19)

This expression can be simplified to give

\[
y(t) = y_0 + \frac{v_{y,0}}{v_{x,0}} (x(t) - x_0) - \frac{1}{2} \frac{g}{v_{x,0}^2} (x(t)^2 - 2 x(t) x_0 + x_0^2).
\]  

(3.8.20)

This is seen to be an equation for a parabola by rearranging terms to find

\[
y(t) = -\frac{1}{2} \frac{g}{v_{x,0}^2} x(t)^2 + \left( \frac{g x_0}{v_{x,0}^2} + \frac{v_{y,0}}{v_{x,0}} \right) x(t) - \frac{v_{y,0}}{v_{x,0}} x_0 - \frac{1}{2} \frac{g}{v_{x,0}^2} x_0^2 + y_0.
\]  

(3.8.21)

The graph of \( y(t) \) as a function of \( x(t) \) is shown in Figure 3.17.
Note that at any point \((x(t), y(t))\) along the parabolic trajectory, the direction of the tangent line at that point makes an angle \(\theta\) with the positive \(x\)-axis as shown in Figure 3.17. This angle is given by

\[
\theta = \tan^{-1}\left(\frac{dy}{dx}\right),
\]

where \(\frac{dy}{dx}\) is the derivative of the function \(y(x) = y(x(t))\) at the point \((x(t), y(t))\).

The velocity vector is given by

\[
\vec{v}(t) = \frac{dx(t)}{dt}\mathbf{i} + \frac{dy(t)}{dt}\mathbf{j} = v_x(t)\mathbf{i} + v_y(t)\mathbf{j}
\]

The direction of the velocity vector at a point \((x(t), y(t))\) can be determined from the components. Let \(\phi\) be the angle that the velocity vector forms with respect to the positive \(x\)-axis. Then

\[
\phi = \tan^{-1}\left(\frac{v_y(t)}{v_x(t)}\right) = \tan^{-1}\left(\frac{dy(t)}{dx(t)}\right) = \tan^{-1}\left(\frac{dy}{dx}\right).
\]

Comparing our two expressions we see that \(\phi = \theta\); the slope of the graph of \(y(t)\) vs. \(x(t)\) at any point determines the direction of the velocity at that point. We cannot tell from our graph of \(y(t)\) how fast the body moves along the curve; the magnitude of the velocity cannot be determined from information about the tangent line.
If, as in Figure 4.16, we choose our origin at the initial position of the body at \( t = 0 \), then \( x_0 = 0 \) and \( y_0 = 0 \). Our orbit equation, Equation (3.8.21) can now be simplified to

\[
y(t) = -\frac{1}{2} \frac{g}{v_{x,0}^2} x(t)^2 + \frac{v_{y,0}}{v_{x,0}} x(t).
\] (3.8.25)

**Part C: Non-Constant Acceleration**

**3.9 Friction as a linear function of velocity.**

We have seen an example where the acceleration of an object was a given non-constant function of time, Equation (3.5.15). In many physical situations the force on an object will be modeled as depending on the object’s velocity. (Forces will be discussed in Chapter 4 and will be indispensable in the rest of this subject.) Some friction models result in an acceleration that is proportional to the velocity,

\[
\ddot{a} = -\gamma \dot{v}.
\] (3.9.1)

where \( \gamma \) is a constant that depends on the properties (density, viscosity) of the medium and the size and shape of the object. Note that \( \gamma \) has dimensions of inverse time,

\[
\text{dim}[\gamma] = \frac{\text{dim}[\text{acceleration}]}{\text{dim}[\text{velocity}]} = \frac{\text{L} \cdot \text{T}^{-2}}{\text{L} \cdot \text{T}^{-1}} = \text{T}^{-1}.
\] (3.9.2)

The minus sign in Equation Error! Reference source not found. indicates that the acceleration is directed opposite to that of the object’s velocity (relative to the fluid). The acceleration has no component perpendicular to the velocity, and in the absence of other forces an object with this acceleration will move in a straight line, but with varying speed. Denote the direction of this motion as the \( x \)-direction, so that Equation (3.9.1) becomes

\[
a_x = \frac{dv_x}{dt} = -\gamma v_x.
\] (3.9.3)

Equation (3.9.3) is now a differential equation. For our purposes, we’ll create an initial-condition problem by specifying that the initial \( x \)-component of velocity is \( v_x(t = 0) = v_{x,0} \).

Two methods of solving this problem, both used by physicists, will be presented here.
The first is the use of an ansatz\(^2\); from Equation (3.9.3) we would expect a graph of \(v\) as a function of time to start at \(v_{x_0}\) with a negative slope, and since \(v\) is decreasing we expect the slope to decrease.

There are many such functions. What we do for an ansatz is to pick a specific functional form that has the desired shape and see how, if at all, the function can be made to satisfy Equation (3.9.3) and the initial condition \(v(t = 0) = v_{x_0}\).

Two such functions are considered here:

\[
\begin{align*}
   v_1 &= v_0 e^{-t/\tau_1}, \\
   v_2 &= v_0 \frac{1}{1 + t/\tau_2},
\end{align*}
\]

where \(\tau_1\) and \(\tau_2\) are constants with dimensions of time that may have to be determined.

The two functions \(v_1\) and \(v_2\) are plotted in Figure 3.18 below. For Plotting purposes, the vertical scale is the ratio \(v_1/v_0\) or \(v_2/v_0\) and the horizontal scale is \(t = t/\tau\). The upper plot (the green plot if viewed in color) is \(v_2\) and the lower (red) is \(v_1\). It should be clear that both plots have the desired qualitative property of decreasing with decreasing slope.

We still need to see which of either of the expressions in (3.9.4) satisfy both Equation (3.9.3) and the initial condition \(v(t = 0) = v_{x_0}\). Both satisfy the initial condition, and indeed the leading factor could be changed as desired to match any initial condition. Performing the differentiations,

\[
\begin{align*}
   \frac{d v_1}{dt} &= -v_0 \frac{1}{\tau_1} e^{-t/\tau_1} = \frac{1}{\tau_1} v_1 \\
   \frac{d v_2}{dt} &= -v_0 \frac{1}{\tau_2} \left(1 + t/\tau_2\right) = \frac{1}{v_0 \tau_2} v_2^2.
\end{align*}
\]

Thus, \(v_2\) as given in the second expression in (3.9.4) cannot be a solution to (3.9.3).

We see that \(v_1\) will be a solution if we choose \(\tau = 1/\gamma\), with the result

\[
v_x = v_0 e^{-\gamma t}.
\]

---

\(^2\) A mathematical assumption, esp. about the form of an unknown function, which is made in order to facilitate solution of an equation or other problem. Oxford English Dictionary. In other words, an inspired guess.
Note that even though the chosen form for $v_2$ did not work for this problem, we see that if we encountered a similar problem with the magnitude of the acceleration proportional to the square of the speed, we’ve got that one solved, if we remember where we did it. (Such a dependence of friction on speed is known as Newtonian Damping, so we suspect it’s worth knowing.)

The differential equation in (3.9.3) is known as a separable equation, in that the equation may be rewritten as

$$\frac{1}{v_x} \frac{dv_x}{dt} = \frac{d}{dt} \ln[v_x] = -\gamma.$$  \hspace{1cm} (3.9.7)

The integration in this case is simple, leading to

$$\ln[v_x] = -\gamma t + \ln[v_0]$$

$$v_x = v_0 e^{-\gamma t},$$  \hspace{1cm} (3.9.8)

reproducing the result of Equation (3.9.6)

It should be noted that the result in Equation (3.9.7) is sometimes obtained by “cross-multiplying” the expression in (3.9.3) to obtain
\[
\frac{dv_y}{v_y} = -\gamma \, dt
\]  

(3.9.9)

and then integrating both sides with respect to the respective integration variables, to obtain the same result. This is indeed equivalent to making a “change of variables” in the calculus wording.

**3.10 Linear Friction with Gravity**

A common extension of the above example is to have an object falling through the same viscous medium, subject to gravity but no other forces. Taking the positive \( y \)-direction to be downward, the equation of motion becomes

\[
a_y = \frac{dv_y}{dt} = g - \gamma v_y.
\]

(3.10.1)

Note that the expression in Equation (3.10.1) is valid for the vertical velocity directed upwards \( (v_y < 0) \) or downwards \( (v_y > 0) \). Unlike the previous example, there is a preferred direction; we expect that in the limit of long times and no other forces, any object would eventually fall straight down. We will use the expectation to simplify our methods of solution, starting with an ansatz that assumes a terminal velocity (actually, a terminal speed) \( v_{\text{term}} \); the terminal velocity is that for which the acceleration given by Equation (3.10.1) is zero, \( v_{\text{term}} = g / \gamma \) (note that \( v_{\text{term}} \) has dimensions of velocity). At this point, it helps to rewrite Equation (3.10.1) as

\[
\frac{dv_y}{dt} = \gamma \left( v_{\text{term}} - v_y \right).
\]

(3.10.2)

If we have, as before, an initial-value problem, in this case the initial condition being \( v_y(0) = v_{y,0} \), our trial solution will be one that has \( v_y(0) = v_{y,0} \) but which approaches \( v_{\text{term}} \) for large times. From our previous experience, we suspect that a function involving an exponential will be more likely to lead to success than a rational function. So, our trial function will be

\[
v_3 = v_{\text{term}} + (v_0 - v_{\text{term}}) e^{-t/\tau_3}.
\]

(3.10.3)

Performing the differentiation,

\[
\frac{dv_3}{dt} = -\frac{1}{\tau_3} (v_0 - v_{\text{term}}) e^{-t/\tau_3}
\]

\[= -\frac{1}{\tau_3} (v_3 - v_{\text{term}})
\]

(3.10.4)
and we see that $v_3$ is a solution to the problem with the choice $\tau_3 = 1/\gamma$;

$$v_y = v_{\text{term}} + (v_0 - v_{\text{term}}) e^{-\gamma t}. \quad (3.10.5)$$

A plot of the ratio $v_y / v_{\text{term}}$ as a function of $t = \gamma t$ is shown in Figure 3.19, for the four different initial conditions $v_0 = v_{\text{term}}$, $v_0 = 2v_{\text{term}}$, $v_0 = 0$ and $v_0 = -v_{\text{term}}$. Note that the last of these (the blue plot if seen in color) is a situation where the object is initially moving upward.

![Figure 3.19](image)

**Figure 3.19** Falling objects with friction

The success of our ansatz suggests a more direct technique. In Equation (3.10.2), make the substitution $u = v_y - v_{\text{term}}$. Recognizing that $du / dt = dv_y / dt$, that equation becomes

$$\frac{du}{dt} = -\gamma u \quad (3.10.6)$$

and the initial condition becomes $u(t = 0) = v_0 - v_{\text{term}}$. This has been reduced to a problem done previously (the previous section) and we can just quote that result;

$$v_y - v_{\text{term}} = (v_0 - v_{\text{term}}) e^{-\gamma t} \quad (3.10.7)$$
and we’re done.

If we hadn’t seen to do this, Equation (3.10.2) is still separable,

$$\frac{1}{v_{\text{term}} - v_y} \frac{dv_y}{dt} = -\gamma \ln[v_{\text{term}} - v_y] = -\gamma . \quad (3.10.8)$$

Integrating and exponentiating, including the initial condition,

$$\ln[v_{\text{term}} - v_y] = -\gamma t + \ln[v_{\text{term}} - v_0]$$

$$v_{\text{term}} - v_y = (v_{\text{term}} - v_0) e^{-\gamma t}, \quad (3.10.9)$$

reproducing the previous result.

In the result, the behavior of the solution should be checked in the large and small limits of time. From the graphs, or from the analytic form, \( v_y \rightarrow v_{\text{term}} \) as \( t \rightarrow \infty \). For small times \( t \ll 1/\gamma \), consider the solution expressed as

$$v_y = v_{\text{term}} (1 - e^{-\gamma t}) + v_0 e^{-\gamma t}$$

$$= \frac{g}{\gamma} (1 - e^{-\gamma t}) + v_0 e^{-\gamma t}$$

$$= \frac{g}{\gamma} (\gamma t) + v_0$$

$$= gt + v_0 \quad (3.10.10)$$

to zero order in \( \gamma \), as expected.