Chapter 7 Appendix: Worked Examples for Work Integrals

In the Course Notes, we considered work integrals only for situations where the displacement, typically \( \Delta \mathbf{x} \) or \( \Delta \mathbf{r} \), was parallel or antiparallel to the net force on an object. It was mentioned that situations in which this simplification was not the case might be encountered, and the full expressions,

\[
dW = \mathbf{F} \cdot d\mathbf{r} = \left( F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \right) \left( dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \right)
\]

\[
= F_x \, dx + F_y \, dy + F_z \, dz
\]

\[
W = \int_{r=r_0}^{r=r_f} \mathbf{F} \cdot d\mathbf{r} = \int_{r=r_0}^{r=r_f} \left( F_x \, dx + F_y \, dy + F_z \, dz \right)
= \int_{r=r_0}^{r=r_f} F_x \, dx + \int_{r=r_0}^{r=r_f} F_y \, dy + \int_{r=r_0}^{r=r_f} F_z \, dz
\]

(7.A.1)

would have to be used, and the possibility

\[
F_x = F_x(x, y, z), \quad F_y = F_y(x, y, z), \quad F_z = F_z(x, y, z)
\]

(7.A.2)

(7.A.3)

could not be ruled out. (The above expressions are Equations (7.7.6), (7.7.7) and (9.7.11) from the notes.)

As a concrete example, consider a force \( \mathbf{F} = \mathbf{F}_{\text{rotational}}(x, y) \) given by

\[
\mathbf{F}_{\text{rotational}} = \frac{F_0}{r_0} (y \mathbf{j} - x \mathbf{i})
\]

(7.A.4)

where \( F_0 \) and \( r_0 \) are positive constants with dimensions of force and length respectively. A sketch of this force (known as a force field) is given below in Figure 7.A.1, with \( r_0 = 1 \) for graphing purposes.

The length of the arrow at any point is proportional to the magnitude of the force, and the direction of the force is indicated by the direction of the arrow. As can be seen, the direction of the force is tangent to a circle centered at the origin. A quick calculation shows that

\[
\mathbf{F}_{\text{rotational}} \cdot \mathbf{r} = \frac{F_0}{r_0} \left( y \mathbf{j} - x \mathbf{i} \right) \cdot \left( x \mathbf{i} + y \mathbf{j} \right) = 0
\]

(7.A.5)

and hence \( \mathbf{F}_{\text{rotational}} \) at a point \((x, y)\) is perpendicular to the vector from the origin to that point.
The purpose of the following calculations of line integrals is to demonstrate two properties:

- The line integral between two points may depend on the path chosen.
- For a given path, the integral is independent of the parameterization of the path. In mechanics terms, we could say that the line integral is independent of the speed with which a body would traverse the path.

Of course, a rigorous generalization of the above would require certain mathematical restrictions regarding continuity and smoothness of the fields and paths. For now, we’ll let these lie.

Consider three paths in the $x$-$y$ plane from the point $(x_0, y_0) = (R, 0)$ to the point $(x_f, y_f) = (0, R)$, $R > 0$:

I. Counterclockwise along the circle $x^2 + y^2 = R^2$.

II. Along the line $x + y = R$.

III. Along the $x$-axis from $(R, 0)$ to the origin $(0,0)$ and the along the $y$-axis to $(0,R)$.
In Figure 7.A.2 below, \( R = 1 \) for plotting purposes. The circular curve (green if viewed in color) is path I and the diagonal line (red) is path II. Path III, along the axes, is not included in the figure.

**Path I – Method 1**

The straightforward “brute force” method is to use \( x \) as the parameter to describe the curve. That is, for any \( 0 \leq x \leq 1 \), there is a corresponding point \( y = y(x) = \sqrt{R^2 - x^2} \) on the circle in the first quadrant. The differential elements \( dx \) and \( dy \) are related by

\[
\frac{dy}{dx} = \frac{-x}{\sqrt{R^2 - x^2}}. \tag{7.A.6}
\]

Before going on, note that the result of (7.A.6) could have been obtained by implicit differentiation:

\[
x^2 + y^2 = R^2
\]
\[
2x + 2y \frac{dy}{dx} = 0. \tag{7.A.7}
\]

From either calculation, \( dx \) and \( dy \) have opposite signs, as suggested in the figure. In the limit as \( y \to 0 \), \( dy/dx \to \infty \), characteristic of a vertical slope, again in agreement with the figure.
We then have
\[
\mathbf{F}_{\text{rotational}} \cdot d\mathbf{r} = \frac{F_0}{r_0} \left( -y \mathbf{i} + x \mathbf{j} \right) \left( dx \mathbf{i} + dy \mathbf{j} \right)
\]
\[
= \frac{F_0}{r_0} \left( -y \, dx + x \, dy \right)
\]
\[
= \frac{F_0}{r_0} \left[ -\sqrt{R^2 - x^2} + \frac{x}{\sqrt{R^2 - x^2}} \right] dx
\]
\[
= \frac{F_0}{r_0} \left[ -\frac{R^2}{\sqrt{R^2 - x^2}} \right] dx.
\] (7.A.8)

The line integral is then
\[
\int_{r=r_0}^{r=r_f} \mathbf{F}_{\text{rotational}} \cdot d\mathbf{r} = -\frac{F_0}{r_0} R^2 \int_0^1 \frac{dx}{\sqrt{R^2 - x^2}} = \frac{F_0}{r_0} R^2 \int_0^1 \frac{dx}{\sqrt{R^2 - x^2}}.
\] (7.A.9)

The last integral is well-known or may be found in standard tables and is equal to \(\pi / 2\), so that
\[
\int_{\text{path } I} \mathbf{F}_{\text{rotational}} \cdot d\mathbf{r} = \frac{\pi R^2 F_0}{2 r_0}.
\] (7.A.10)

Note that this result has dimensions of \([\text{force}] \cdot [\text{distance}] = [\text{energy}]\), as it must.

**Path I – Method 2**

The circular arc strongly suggests plane polar coordinates. Parameterize the Cartesian coordinates by
\[
x(t) = R \cos \omega t
\]
\[
y(t) = R \sin \omega t
\] (7.A.11)

for \(\omega > 0\), \(0 \leq t \leq \pi / 2\). The differentials are straightforward,
\[
dx = -\omega R \sin \omega t, \quad dy = \omega R \cos \omega t
\] (7.A.12)
\[
- y \, dx + x \, dy = R^2 \omega \left( \sin^2 \omega t + \cos^2 \omega t \right) dt = R^2 \omega \, dt.
\] (7.A.13)

The integral is now trivial,
\[ \int_{\text{path I}} \vec{F}_{\text{rotational}} \cdot d\vec{r} = \frac{F_0}{r_0} R^2 \omega \int_0^{\pi/2} dt = \frac{\pi R^2 F_0}{2r_0} \]  

(7.A.14)

as found by Method I.

Note that the result in (7.A.14) does not depend on the parameter \( \omega \); the work integral is independent of “how fast” the arc is traversed.

For those interested in the details of the calculus involved, the last integral in Equation (7.A.9) is most commonly done by making the substitution \( x = R \cos u \) or \( x = R \sin u \), resulting in an integral identical (except for the presence of \( \omega \)) to that in (7.A.14).

Path II – Any Method

Along the diagonal line, \( dx = -dy \), either by inspection or by differentiation. With \( y = R - x \),

\[ -y \, dx + x \, dy = -(R - x) \, dx + x(-dx) = -R \, dx \]  

(7.A.15)

with the result that

\[ \int_{\text{path II}} \vec{F}_{\text{rotational}} \cdot d\vec{r} = \frac{F_0}{r_0} \int_R^0 (-R) \, dx = \frac{R^2 F_0}{r_0} \cdot (\text{7.A.16}) \]

Path III – No Method Needed

On the \( x \)-axis the \( x \)-component of \( \vec{F}_{\text{rotational}} \) vanishes, and in moving along the axis, \( \vec{F}_{\text{rotational}} \cdot d\vec{r} = 0 \). This is seen either from the algebraic form of \( \vec{F}_{\text{rotational}} \) or from either of the above sketches. Similarly, \( \vec{F}_{\text{rotational}} \cdot d\vec{r} = 0 \) when moving along the \( y \)-axis. The very clean result is

\[ \int_{\text{path III}} \vec{F}_{\text{rotational}} \cdot d\vec{r} = 0 \cdot \]  

(7.A.17)

The fact is, the force field \( \vec{F}_{\text{rotational}} \) was rigged to allow easy results. Maybe you’ve seen the pattern; the line integral is twice the area between the curve and the positive axes. A geometric interpretation is that \(-y \, dx\) is the area contained in a strip of height \( y \) and width \(|dx|\) for negative \( dx \), as was the case for all three paths. Similarly, \( x \, dy \) is the area of a strip of width \( x \) and height \( dy \) for positive \( dy \), again as was the case for all three paths. This is likely to be explained in more detail, and almost certainly with different notation, in your 18.02 class. Remember where you saw it first.
Why “rotational”? If instead of a force, we had a velocity field

$$\mathbf{V}_{\text{rotational}} = \frac{V_0}{r_0}(-y\hat{i} + x\hat{j}),$$  \hspace{1cm} (7.A.18)

this would be the velocity of a point on a disc spinning about the $z$-axis with angular velocity $\omega_0 = V_0 / r_0$. In fact,

$$\mathbf{V}_{\text{rotational}} = \omega_0(-y\hat{i} + x\hat{j}) = \omega_0 \hat{k} \times \mathbf{r},$$  \hspace{1cm} (7.A.19)

an important and convenient way to express the rotational motion of rigid bodies.

If we had a variation

$$\mathbf{V}_{\text{wind}} = \Omega(r)(-y\hat{i} + x\hat{j}),$$  \hspace{1cm} (7.A.20)

where $\Omega(r)$ is some function of the distance $r$ with dimensions of angular frequency, we might have a model of a cyclone or hurricane (but some variation with altitude would certainly be needed).