Chapter 15 Appendix
Moment of Inertia of a Spherical Shell

It is common to regard the rotation of a rigid object with spherical symmetry; we live on one such object (not entirely uniform and not entirely rigid, and not even perfectly spherical). This Appendix will show how to calculate the moment of inertia of such an object about an axis passing through the center of mass.

Part of the goal of this Appendix is to show that such calculations, while far from trivial, are well within the scope of an 18.01-level calculus subject at MIT.

Accordingly, the presentation is not the standard one. We will first find the moment of inertia of a solid sphere, in two different but equivalent methods, and from that result find the moment of inertia of a thin spherical shell. The result for a shell will then be used to find the moment of inertia of a spherical shell that may not be thin, leading to an expression that can be generalized to spherically symmetric mass distributions of varying mass densities.

In what follows, it will be important to match the notation to the situation under consideration. Specifically, the symbol \( dI \) or \( \Delta I \) for the differential or small contribution of a mass element \( dm \) or \( \Delta M \) will be used often, and in different circumstances. In all cases except the last application, the object under consideration will have uniform mass density \( \rho \). All moments of inertia will be found with respect to an axis passing through the center of mass, which is the geometric center for a uniform spherically symmetric object. For convenience of calculation, this axis will be taken to be the \( z \)-axis and the center of mass will be at the coordinate \( z = 0 \).

**Moment of Inertia of a Uniform Solid Sphere – Disc Method**

Denote the radius of the sphere by \( R \), and as explained above, the constant mass density is denoted by \( \rho \). The use of the term “Disc Method” is a reference to a method of calculation of volumes often presented in 18.01 at MIT.

Our strategy will be to divide the sphere into infinitesimally small discs with axes in the \( z \)-direction, thickness \( dz \) and radii \( r \). Accordingly, the center of each disc will be located at the coordinate \( z \) from the origin and each disc will have a radius \( r = \sqrt{R^2 - z^2} \) and an infinitesimal contribution \( dl \) to the moment of inertia

\[
dl = \frac{1}{2} (dm) r^2 = \frac{1}{2} (\rho \pi r^2 dz) r^2 \\
= \frac{1}{2} \rho \pi r^4 \ dz = \frac{1}{2} \rho \pi (R^2 - z^2)^2 \ dz.
\] (15.A.1)
The needed integral suggested by the last expression in (15.A.1) is hardly formidable. Expanding the square of the term in parentheses,

\[ dI = \frac{1}{2} \rho \pi \left( R^4 - 2R^2z^2 + z^4 \right) dz. \]  

(15.A.2)

The limits on the \( z \)-integral are from \(-R\) to \(R\), and so

\[
I = \int dI = \frac{1}{2} \rho \pi \left[ R^4 z - \frac{2}{3} R^2 z^3 + \frac{1}{5} z^5 \right]_R^{\infty} = \rho \pi R^5 \left[ 1 - \frac{2}{3} + \frac{1}{5} \right] = \rho \pi R^5 \left[ \frac{8}{15} \right].
\]  

(15.A.3)

(Some minor algebraic steps have been skipped in the above calculation.)

The mass of the sphere is the product of the volume and the density,

\[ M = \frac{4}{3} \pi R^3 \rho, \]  

(15.A.4)

and combining the results of (15.A.3) and Equation (15.A.4), we see that

\[ I = \frac{2}{5} M R^2. \]  

(15.A.5)

**Moment of Inertia of a Uniform Solid Sphere – Shell Method**

As for the Disc Method, denote the radius and density of the sphere by \( R \) and \( \rho \). The use of the term “Shell Method” is also a reference to a method of calculation of volumes often presented in 18.01 at MIT.

For the Shell Method, divide the sphere into coaxial right circular cylinders (“shells”) of radius \( r \), length \( 2z \) and thickness \( dr \). The differential moment of inertia of each shell with respect to the origin about the \( z \)-axis is

\[ dI = r^2 dm. \]  

(15.A.6)

The differential mass \( dm \) is the product of the differential volume and the mass density,

\[ dm = (2\pi r)(2z)dz \rho. \]  

(15.A.7)

To express \( r \) as a function of \( z \), we use the same relation as in the disc method,
\[ r = \sqrt{R^2 - z^2} \]
\[ z = \sqrt{R^2 - r^2} \]

so that the infinitesimal contribution to the moment of inertia is

\[ dl = 4\pi\rho r^3 \sqrt{R^2 - r^2} \, dr \]
\[ = (2\pi\rho) r^2 \sqrt{R^2 - r^2} (2r \, dr). \]

The last manipulation in (15.A.9) was made in anticipation of making the substitution

\[ u = R^2 - r^2 \]
\[ du = -2(r \, dr), \]

which leads to

\[ dl = -2\pi\rho (R^2 - u)^{3/2} \, du \]
\[ = -2\pi\rho (R^2 u^{1/2} - u^{3/2}) \, du. \]

The limits on the definite integral using the substitution in (15.A.10) are from \( u = R^2 \) (when \( r = 0 \)) to \( u = 0 \) (when \( r = R \)). Upon integration,

\[ I = \int dl = (-2\pi\rho) \left[ R^2 \frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right]_0^{R^2} \]
\[ = 2\pi\rho R^3 \left[ \frac{4}{15} \right], \]

reproducing the result in (15.A.3).

We have sort of pulled a fast one. If in the second equation in (15.A.9) we had made a different substitution, \( z' = \sqrt{R^2 - r^2} \), we would reproduce the definite integral in (15.A.2) right away.

**Moment of Inertia of a Thin Shell**

For a uniform solid sphere of mass \( M \) and radius \( R \), we have found that the moment of inertia about any axis passing through the origin,

\[ I = \frac{2}{5} M R^2. \]
Suppose that a thin layer of the same mass density and mass $\Delta M$ is added to this sphere, increasing the sphere’s radius by $\Delta R$. Expressing the moment of inertia in terms of the density and the radius, as in Equations (15.A.3) and (15.A.12),

$$I = \frac{8}{15} \pi \rho R^5$$  \hspace{1cm} (15.A.14)

the small change $\Delta I$ in the moment of inertia is given by

$$\Delta I = \frac{8}{15} \pi \rho \left( 5 R^4 \Delta R \right)$$

$$= \frac{2}{3} \left( 4 \pi \rho R^2 \Delta R \right) R^2.$$  \hspace{1cm} (15.A.15)

The term in parentheses in the second expression in (15.A.15) is the added mass $\Delta M$ and thus the moment of inertia of a thin-walled spherical shell in terms of the mass and radius is, using the more conventional $M$ instead of $\Delta M$ as the mass of the shell,

$$I = \frac{2}{3} M R^2.$$  \hspace{1cm} (15.A.16)

**Moment of Inertia of a Spherical Shell of Arbitrary Thickness**

Re-express the first expression in (15.A.15) as a differential,

$$dI = \frac{8}{3} \pi \rho r^4 \, dr.$$  \hspace{1cm} (15.A.17)

The moment of inertia of a shell of inner radius $a$ and outer radius $b$ is then found by straightforward integration of a polynomial to be

$$I = \frac{8}{15} \pi \rho \left( b^5 - a^5 \right)$$  \hspace{1cm} (15.A.18)

An even simpler integration, or considering the difference between the volumes of two spheres, gives the mass as

$$M = \frac{4}{3} \pi \rho \left( b^3 - a^3 \right)$$  \hspace{1cm} (15.A.19)

so that the moment of inertia in terms of the mass is

$$I = \frac{2}{5} M \frac{b^5 - a^5}{b^3 - a^3}.$$  \hspace{1cm} (15.A.20)
Note that if \( a = 0 \) and \( b = R \), Equation (15.A.13) is recovered. For arbitrary \( a \) and \( b \), Equation (15.A.20) may or may not be the most useful form. By noting that

\[
\begin{align*}
    b^5 - a^5 &= (b-a)(b^4 + b^3a + b^2a^2 + ba^3 + a^4) \\
    b^3 - a^3 &= (b-a)(b^2 + ba + a^2),
\end{align*}
\]

Equation (15.A.21) becomes

\[
I = \frac{2}{5}M \frac{b^4 + b^3a + b^2a^2 + ba^3 + a^4}{b^3 + ba + a^2},
\]

which, while still cumbersome, does reproduce Equation (15.A.16) for \( a = b = R \).

**Moment of Inertia for a Sphere with Varying Density**

In many circumstances, the mass density of an object with spherical symmetry could vary with distance from the origin. Planets and stars are often represented as having spherical symmetry, but the mass densities are only approximately constant in special cases.

If the density is then \( \rho = \rho(r) \), Equation (15.A.17) becomes

\[
dI = \frac{8}{3}\pi \rho(r)r^4 dr
\]

which is then integrated from the origin to an outer radius \( R \) to find

\[
I = \frac{8}{3}\pi \int_0^R \rho(r)r^4 dr.
\]

To express the moment of inertia in terms of the total mass, a similar procedure is used to find

\[
M = 4\pi \int_0^R \rho(r)r^2 dr.
\]

Combining Equations (15.A.24) and (15.A.25) leads to

\[
I = \frac{2}{3}M \frac{\int_0^R \rho(r)r^4 dr}{\int_0^R \rho(r)r^2 dr},
\]
a result that cannot be simplified unless \( \rho(r) \) is known or known to have a fairly simple
dependence on radius, and in the above examples where \( \rho(r) \) was either a constant or
zero (for the inside of a shell).