Correction and Further Explanation

Today (Wednesday, February 26), in trying to do something sort of simple the hard way, I made a mistake (now there’s a surprise). The part done correctly was that if

\[ V_1 = V_0 \left( x^2 - y^2 \right) \]

where \( V_0 \) is a constant, then (i) \( \nabla^2 V_1 = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) V_1 = 0 \), (ii) the equipotentials of \( V_1 \) are hyperbolae with asymptotes \( y = \pm x \), (iii) The equipotential \( V_1 = 0 \) is the intersection of those asymptotes and (iv) the electric field lines (sometimes known as the integral curves of \( \vec{\nabla} V_1 = \vec{0} \)) are tangent to the hyperbolae \( xy = \text{constant} \), with asymptotes \( x = 0, y = 0 \).

The wrong part (not wise to try, and a mistake that has fortunately gone the way of all chalk) was to find another function with the same \( V = 0 \) equipotentials but with \( V \) falling off with distance from the origin. I won’t give the derivation, but the function we want is

\[ V_2 = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \]

which is singular at the origin. When you have lots of free time, show that indeed \( \nabla^2 V_2 = 0 \) everywhere except the origin.

For another way to find the form of \( V_2 \), which I had misguessed in my head, is that within a constant multiple,

\[ V_2(x, y) = \left( \hat{i} + \hat{j} \right) \cdot \vec{\nabla} \left[ \left( \hat{i} - \hat{j} \right) \cdot \vec{\nabla} \left( \ln \left( x^2 + y^2 \right) \right) \right], \]

which is a really neat result that I’d be delighted to show you, but goes well beyond the usual bounds of 8.02.

A countour map of \( V_2(x, y) \) is shown.