Supplemental Notes

Herein are the details of the derivation of the torque on a current loop. This is for the case of a uniform magnetic field.

What we wish to find is
\[ \oint \mathbf{r} \times \left( I \, d\mathbf{r} \times \mathbf{B} \right), \]
where \( \mathbf{B} \) is a constant. What we do is to first consider the vector differential
\[ d \left( \mathbf{r} \times \left( \mathbf{r} \times \mathbf{B} \right) \right) = d\mathbf{r} \times \left( \mathbf{r} \times \mathbf{B} \right) + \mathbf{r} \times \left( d\mathbf{r} \times \mathbf{B} \right), \]
from which
\[ \oint d\mathbf{r} \times \left( \mathbf{r} \times \mathbf{B} \right) + \oint \mathbf{r} \times \left( d\mathbf{r} \times \mathbf{B} \right) = \mathbf{0}. \]

Now we use a neat fact about cross products that will be stated without proof, namely that
\[ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}, \]
which we use to get
\[ \oint \mathbf{r} \times \left( d\mathbf{r} \times \mathbf{B} \right) + \oint \mathbf{r} \times \left( \mathbf{B} \times d\mathbf{r} \right) + \oint \mathbf{B} \times (\mathbf{r} \times d\mathbf{r}) = \mathbf{0}. \]

Now for the fun; in the above, switch the order of \( \mathbf{B} \) and \( \mathbf{r} \) in the second integral, move the third integral to the right side, move \( \mathbf{B} \) out of the integral and then flip \( \mathbf{B} \) to the right of the integral. Use the first result above to equate to second integral to the first to obtain
\[ 2 \oint \mathbf{r} \times \left( d\mathbf{r} \times \mathbf{B} \right) = \left( \oint \mathbf{r} \times d\mathbf{r} \right) \times \mathbf{B}. \]

Multiply by the current \( I \) and divide by 2 to get our neat result
\[ \mathbf{\tau} = \oint \mathbf{r} \times \left( I \, d\mathbf{r} \times \mathbf{B} \right) = \left( I \oint d\mathbf{A} \right) \times \mathbf{B} = \mu \times \mathbf{B}, \]
where the fact that
\[ d\mathbf{A} = \frac{1}{2} \mathbf{r} \times d\mathbf{r}. \]
has been used.

This really is legitimate; what we have done is an integral by parts of a vector function, evaluating the integrand at the same points in space. The factor of two is equivalent to that which comes from integrating $x\,dx$ by parts;

$$\int x\,dx = x^2 - \int dx\,x = x^2 - \int x\,dx,$$

from which $\int x\,dx = x^2/2$.

Note that in the above, it was crucial to have $\vec{B}$ be uniform. Also, note that $d\vec{r}$ has been used instead of $d\vec{l}$ and that the area element is $d\vec{A}$ instead of $d\vec{S}$. While we’re noting things, note that at no point did we need to assume that the loop was planar. This could lead to things that are even more fun.