Problem 1 Ampere’s Law and Force on a Wire Loop

A rectangular wire loop of length $s$ and width $w$ carries a current $I_1$ and located near a very long straight wire (you may assume it is an infinite wire) carrying current $I_2$ that is fixed in place. The lower leg of the wire loop is a distance $a$ from the long straight wire. The directions of the currents are shown in the figure above.

a) Use Ampere’s Law to find a vector expression for the magnitude and direction of the magnetic field of the very long straight wire (you may assume it is an infinite wire) carrying a current $I_2$ at the point $P$, which is located a positive distance $y$ from the wire. Express your answer in terms of $\mu_0$, $I_2$, $y$, $\hat{i}$, $\hat{j}$, and $\hat{k}$ as needed. You must show all your work in order to receive credit.
b) At the instant depicted in the figure below, determine the direction and magnitude of the magnetic force on each leg of the wire loop. Express your answer in terms of $\mu_0$, $I_1$, $I_2$, $a$, $s$, $w$, $\hat{i}$, $\hat{j}$, and $\hat{k}$ as needed. **You must show all your work in order to receive credit.**

![Diagram of a wire loop with currents and forces](image)

\[ I_1 \]
\[ I_2 \]
\[ s \]
\[ w \]
\[ a \]
\[ \hat{i} \]
\[ \hat{j} \]
\[ \hat{k} \]

\[ \mu_0 \]
\[ I_1 \]
\[ I_2 \]
\[ a \]
\[ s \]
\[ w \]
\[ \hat{i} \]
\[ \hat{j} \]
\[ \hat{k} \]

**c)** Using your results from part b), at the instant depicted in the figure above, determine the direction and magnitude of the total magnetic force on the wire loop. Express your answer in terms of $\mu_0$, $I_1$, $I_2$, $a$, $s$, $w$, $\hat{i}$, $\hat{j}$, and $\hat{k}$ as needed. **You must show all your work in order to receive credit.**
Problem 1: Solution:

a) Ampere’s Law:

\[ B_z 2\pi y = \mu_0 I_2 \Rightarrow \vec{B}_z = \frac{\mu_0 I_2}{2\pi y} \hat{k}. \]

b) Force on left leg (leg a):

\[
\vec{F}_a = \int_a^{a+w} I_1 d\vec{s} \times \vec{B}_2 = \int_a^{a+w} I_1 dy \hat{j} \times \frac{\mu_0 I_2}{2\pi y} \hat{k} = \int_a^{a+w} I_1 dy \frac{\mu_0 I_2}{2\pi y} \hat{i}
\]

\[
\vec{F}_a = \frac{\mu_0 I_2 I_1}{2\pi} \ln(a + w/a) \hat{i}.
\]

Force on right leg (leg c) is opposite force on leg a:

\[
\vec{F}_c = -\frac{\mu_0 I_2 I_1}{2\pi} \ln(a + w/a) \hat{i}.
\]

Force on top leg (leg b):

\[
\vec{F}_b = I_1 \hat{l}_b \times \vec{B}_2(b) = I_1 s \hat{i} \times \frac{\mu_0 I_2}{2\pi (w + a)} \hat{k} = \frac{\mu_0 I_2 I_1 s}{2\pi (w + a)} (-\hat{j}).
\]

Force on bottom leg (leg d):

\[
\vec{F}_d = I_1 \hat{l}_d \times \vec{B}_2(d) = I_1 s (-\hat{i}) \times \frac{\mu_0 I_2}{2\pi a} \hat{k} = \frac{\mu_0 I_2 I_1 s}{2\pi a} \hat{j}.
\]

(c) Total force:

\[
\vec{F}^T = \vec{F}_b + \vec{F}_d = \frac{\mu_0 I_2 I_1 s}{2\pi} \left( \frac{1}{a} - \frac{1}{w + a} \right) \hat{j}.
\]
Problem 2 Torque on a Dipole

A ring of radius $R$, located in the $xy$-plane and centered about the $z$-axis, carries a current $I$ directed clockwise as seen from above, (Figure (a)).

a) Determine the direction and magnitude of the magnetic field at a point $P$ that lies on the positive $z$-axis a distance $z$ from the origin. Express your answers in terms of $R$, $\mu_0$, $z$, $I$, and $\hat{k}$ as needed.

b) A point-like magnetic dipole is now placed at the point $P$. The dipole moment is given by $\vec{\mu} = \mu_z \hat{k}$, where $\mu_z > 0$, (Figure (b)). The dipole is not allowed to rotate but can move in the $z$-direction.

(i) What is the direction of the magnetic force on the dipole? Draw the direction of the force on the figure above right.

(ii) Determine an expression for the magnitude of the force on the dipole. Express your answer in terms of $R$, $\mu_0$, $z$, $I$, and $\mu_z$ as needed.
Solution:

(a) First apply the Biot-Savart Law:

\[
\mathbf{B}(P) = \int_{\text{wire}} d\mathbf{B} = \int_{\text{wire}} \frac{\mu_0 I}{4\pi} \frac{d\mathbf{s} \times \hat{r}_{ds,P}}{r_{ds,P}^2}
\]

In polar coordinates: \(Id\mathbf{s} = -IRd\theta \hat{\hat{\theta}}, \hat{r}_{ds,P} = \cos \phi \hat{k} - \sin \phi \hat{r}, r_{ds,P}^2 = R^2 + z^2\). Therefore:

\[
\mathbf{B}(P) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{-IRd\theta \hat{\hat{\theta}} \times (\cos \phi \hat{k} - \sin \phi \hat{r})}{(R^2 + z^2)} (R^2 + z^2)
\]

The horizontal component cancels upon integration; consider pairs of current elements on opposite sides of the circular wire. Factor for direction)

\[
\mathbf{B}(P) = -\frac{\mu_0 IR \sin \phi}{2(R^2 + z^2)} \hat{k}
\]

From the geometry:

\[
\sin \phi = \frac{R}{(R^2 + z^2)^{1/2}}
\]

Therefore the magnetic field at the point \( P \) is:

\[
\mathbf{B}(P) = -\frac{\mu_0 IR^2}{2(R^2 + z^2)^{3/2}} \hat{k}.
\]

(b) The force on the dipole is given by

\[
F_z = \mu_z \frac{\partial B}{\partial z}, \text{ where } \mu_z > 0, \frac{\partial B}{\partial z} > 0 \text{ therefore } F_z = \mu_z \frac{\partial B}{\partial z} > 0.
\]

(c) The derivative of the \( z \)-component of the magnetic field is

\[
\frac{\partial B}{\partial z} = -\frac{3\mu_0 IR^2 z}{2(R^2 + z^2)^{5/2}}.
\]

The magnitude of the force is therefore
\[ |F_z| = |\mu_z| \frac{3\mu_0 IR^2 z}{2(R^2 + z^2)^{3/2}}. \]
Problem 3 Helical Motion of Charged Particle in a Solenoid

A very long solenoid has $n$ turns per unit length, (you may treat the solenoid as infinite in length). Each turn carries a current $I$ as shown in the figures above. The central axis of the solenoid is aligned along the $y$-axis.

At the instant depicted in the figure above left, a positively charged particle, $q > 0$, of mass $m$ is located along the central $y$-axis, traveling with velocity

$$\vec{v} = v_x \hat{i} + v_y \hat{j},$$

where $v_x > 0$ is the component of the velocity in the $x$-direction and $v_y > 0$ is the component of the velocity in the $y$-direction. A view from the perspective of an observer located at the origin looking down the positive $y$-axis of the solenoid is shown in the figure above right. Express your answers to the following questions in terms of $\mu_0$, $n$, $I$, $q$, $m$, $v_x$, $v_y$, $\hat{i}$, $\hat{j}$, and $\hat{k}$ as needed.

a) Use Ampere’s Law to find the direction and magnitude of the magnetic field inside the solenoid due to the current flowing in the solenoid. Be sure to show your Ampèreian loop and all your calculations. **Answers without any work will not receive any credit.**

b) What is the force on the particle at the instant depicted in the figure below? Ignore fringe fields; i.e., consider the field to be uniform.

c) When the particle is inside the solenoid, it makes a helical trajectory i.e. the motion in the $xz$-plane is circular. What is the radius of the circular motion in the $xz$-plane?
d) After the particle makes one helical turn, how far has it traveled in the \( y \)-direction?

**Solution:**

Part (a) Use Ampere’s Law to calculate the magnetic field inside the solenoid yielding

\[
Bl = \mu_0 nIl \Rightarrow \mathbf{B} = \mu_0 nI \hat{j}.
\]

Part (b) The force on the moving charge is given by

\[
\mathbf{F}_q = q\mathbf{v}_q \times \mathbf{B} = q(v_x \hat{i} + v_y \hat{j}) \times \mu_0 nI \hat{j} = qv_y \mu_0 nI \hat{k}.
\]

Part (c): Apply Newton’s Second Law \( \mathbf{F}_q = m\mathbf{a} \) for the particle in the \( \hat{k} \)-direction:

\[
qv_y \mu_0 nI \hat{k} = \frac{mv_y^2}{R} \hat{k}.
\]

Solve for the radius of the orbit:

\[
R = \frac{mv_y}{q\mu_0 nI}.
\]

Part (d): The period of the orbit is

\[
T = \frac{2\pi R}{v_x} = \frac{2\pi m}{q\mu_0 nI}.
\]

Therefore in one period, the particle will travel a distance down the solenoid equal to

\[
\Delta y = v_y T = \frac{2\pi mv_y}{q\mu_0 nI}.
\]
Problem 4 Biot-Savart Law for A Current Loop

Consider a wire consisting of two semi-circles of radii \( R \) and \( 2R \) carrying a current \( I \). Calculate the magnetic field at the center, (point \( P \) in the figure).

Solution: This is very similar to calculating the field of a circular coil instead now we will integrate from 0 to \( \pi \) instead of 0 to \( 2\pi \) to get the contribution from each semi-circular coil. Each semi-circle creates a field in the same direction (out of the page) so we can just superimpose (add) them.

We will use the Biot-Savart Law. Choose cylindrical coordinates with unit vectors \((\hat{\rho}, \hat{\theta}, \hat{k})\). Note \( \hat{\rho} \) is the unit vector that points radially outward in the plane. We choose that notation so as not to get confused with \( \hat{r} = -\hat{\rho} \). We have for the current element \( I d\vec{s} = I r d\theta \hat{\theta} \), for the unit vector from the current element to the field point at the center of the rings, \( \hat{r} = -\hat{\rho} \). Therefore the Biot-Savart Law becomes

\[
d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{(I r d\theta \hat{\theta} \times -\hat{\rho})}{r^2} = \frac{\mu_0 I d\theta}{4\pi} \hat{k}
\]

Integrating for the half ring of radius \( r = 2R \) we get that

\[
\vec{B}_{2R} = \frac{\mu_0 I}{8\pi R} \int_0^\pi d\theta \hat{k} = \frac{\mu_0 I}{8R} \hat{k}.
\]

In a similar fashion, the magnetic field for the half ring of radius \( r = R \) is

\[
\vec{B}_R = \frac{\mu_0 I}{4\pi R} \int_0^\pi d\theta \hat{k} = \frac{\mu_0 I}{4R} \hat{k}.
\]

So the superposition of these two contributions gives the magnetic field at the point \( P \)
\[ \mathbf{\tilde{B}} = \frac{\mu_0 I}{4} \left( \frac{1}{R} + \frac{1}{2R} \right) \mathbf{\hat{k}} = \frac{3\mu_0 I}{8R} \mathbf{\hat{k}} \text{(out of the page)}. \]
Problem 5 Force on a Wire due to Two Current Slabs

The figure above shows two infinite slabs of current. Both slabs of current are infinite in the \(xz\)-directions, and have thickness \(d\) in the \(y\)-direction. The slabs are separated by a distance \(2a\). The top slab of current is located in the region \(0 \leq y \leq d\) and has a constant current density \(\mathbf{J}_{\text{out}} = J \mathbf{z}\) out of the plane of the figure, where \(J > 0\). The bottom slab of current is located in the region \(-(d + 2a) \leq y \leq -2a\) and has a constant current density \(\mathbf{J}_{\text{in}} = -J \mathbf{z}\) into plane of the figure. An infinite wire is placed midway between the two slabs carrying a current \(I\) out of the plane of the figure. Express your answers to the following questions in terms of \(\mu_0\), \(J\), \(a\), \(d\), \(I\), \(\mathbf{i}\), \(\mathbf{j}\), and \(\mathbf{k}\) as needed. **You must show all your work in order to receive credit.**

a) What is the direction and magnitude of the magnetic field due to the two slabs in the region \(y \geq d\)?

b) What is the direction and magnitude of the magnetic field due to the two slabs in the region \(0 \leq y \leq d\)?

c) What is the direction and magnitude of the magnetic field due to the two slabs in the region \(-(d + 2a) \leq y \leq -2a\)?

d) Consider a small length \(ds\) of the current carrying wire. What is the direction and magnitude of the force per unit length \(d\mathbf{F} / ds\) on that small length of wire?
Solution:

Part (a): The magnetic field for the regions \( y \geq d \) and \( y \leq -(2d + a) \) is the superposition of the magnetic fields from the two slabs, \( \vec{B} = \vec{B}_1 + \vec{B}_2 \), where \( \vec{B}_1 \) is the field due to the upper slab and \( \vec{B}_2 \) is the field due to the lower slab.

Apply Ampere’s Law to find the field of each slab. For upper slab, choose an Ampereian loop centered about \( y = d/2 \) with legs in the regions \( y > d \) and \( y < 0 \) and circulate counterclockwise as shown in the figure below.

\[
\int \vec{B} \cdot d\vec{s} = \mu_0 \int \vec{J} \cdot d\vec{A}
\]

Ampere’s Law becomes \( 2B_1l = \mu_0 Jl \). The magnetic field is then

\[
\vec{B}_1 = \begin{cases} 
-\left(\frac{\mu_0 Jd}{2}\right) \hat{i} & y > +d \\
+\left(\frac{\mu_0 Jd}{2}\right) \hat{i} & y < 0 
\end{cases}
\]

A similar calculation for the lower slab with an Ampereian loop as shown in the figure below, circulating clockwise yields

\[
\vec{B}_2 = \begin{cases} 
\left(\frac{\mu_0 Jd}{2}\right) \hat{i} & y > -2a \\
-\left(\frac{\mu_0 Jd}{2}\right) \hat{i} & y < -(2a + d)
\end{cases}
\]

Therefore outside the slabs the sum of the two contributions field is zero.
\[ \vec{B} = \vec{B}_1 + \vec{B}_2 = \begin{cases} 
-(\mu_0 J d / 2) \hat{i} + (\mu_0 J d / 2) \hat{i} = \vec{0}; & y \geq d \\
+(\mu_0 J d / 2) \hat{i} - (\mu_0 J d / 2) \hat{i} = \vec{0}; & y \leq -(2a + d) \end{cases} . \]

Part (b):
Using the results from part (a) for the region, \( 0 \leq y < d \), apply Ampere’s Law to the total field \( \vec{B} \). Choose an Ampereian loop as shown in the figure below (counterclockwise circulation).

Because the total field is zero above the slab, Ampere’s Law becomes
\[ Bl = \mu_0 Jl(d - y) . \]

Therefore the magnetic field in this region is
\[ \vec{B} = (\mu_0 J(d - y)) \hat{i}; \quad 0 \leq y \leq d . \]

Part (c): Again using the results from part (a) for the region, \(- (d + 2a) \leq y \leq -2a\), apply Ampere’s Law to the total field \( \vec{B} \). Choose an Ampereian loop as shown in the figure below (counterclockwise circulation).

Because the total field is zero below the lower slab, Ampere’s Law becomes
\[ Bl = \mu_0 Jl((2a + d) + y) . \]
Therefore the magnetic field in this region is

\[
\mathbf{B} = (\mu_0 J((2a+d)+y)) \hat{i}; \quad -(2a+d) \leq y \leq -2a.
\]

Part (d): The magnetic field in between the slabs is uniform and by the superposition principle and the results from Part (a) is given by

\[
\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = (\mu_0 Jd/2) \hat{i} + (\mu_0 Jd/2) \hat{i} = (\mu_0 Jd) \hat{i}; \quad -(2a) \leq y \leq 0.
\]

The force on a small piece \( ds \) of the wire is then

\[
d\mathbf{F} = I d\hat{s} \times \mathbf{B} = I d\hat{s} \hat{k} \times (\mu_0 Jd) \hat{i} = I d\hat{s} \mu_0 Jd \hat{j}.
\]

Therefore the force per unit length is

\[
\frac{d\mathbf{F}}{ds} = I \mu_0 Jd \hat{j}.
\]
Problem 6 Charge Particle Moving In Uniform Magnetic fields and Across a Potential Difference

A positively charged ion with charge $q$ and mass $m$, initially at rest inside the source, is accelerated across a gap by an electric potential difference with magnitude $\Delta V$. The ion enters a region of uniform magnetic field $B_1$ pointing into the page of the figure above and follows a semi-circular trajectory of radius $R_1$. The ion is again accelerated across the gap, increasing its speed, by an electric potential difference that has the opposite sign but has the same magnitude $|\Delta V|$. The ion then enters a region of uniform magnetic field $B_2$ pointing into the page of the figure above and follows a semi-circular trajectory of radius $R_2 = 2R_1$. What is the ratio $B_2/B_1$ of the magnitudes of the magnetic field in the different regions?
Solution:

Initially the particle is accelerated from rest across a potential difference of magnitude \( \Delta V \). Therefore by the work-energy theorem, the particle attains a kinetic energy

\[
\frac{1}{2}m v_1^2 = q|\Delta V|.
\]

just upon entry of the uniform magnetic field. It then undergoes circular motion. For circular motion in a uniform magnetic field, Newton’s Second Law becomes

\[
q v_1 B_1 = \frac{m v_1^2}{R_1} \Rightarrow R_1 = \frac{m v_1}{q B_1}.
\]

The particle is then accelerated across the same potential difference and by the work energy theorem, the change in kinetic energy is given by

\[
\frac{1}{2}m v_2^2 - \frac{1}{2}m v_1^2 = q|\Delta V|.
\]

Because \( \frac{1}{2}m v_1^2 = q|\Delta V| \), the kinetic energy entering the lower region is then

\[
\frac{1}{2}m v_2^2 = m v_1^2 \Rightarrow v_2 = \sqrt{2} v_1.
\]

The particle then makes a circular orbit with

\[
q v_2 B_2 = \frac{m v_2^2}{R_2} \Rightarrow R_2 = \frac{m v_2}{q B_2}.
\]

The radii of the orbits are related by

\[
R_2 = 2 R_1 \Rightarrow \frac{m v_2}{q B_2} = 2 \frac{m v_1}{q B_1}.
\]

Therefore the ratio of the magnetic fields is

\[
\frac{B_2}{B_1} = \frac{1}{2} \frac{v_2}{v_1} = \frac{\sqrt{2}}{2}.
\]
Problem 7 Time Varying Current Through a Cylindrical Wire

Consider an infinite cylindrical solid wire that has radius $a$. The wire has a time varying current with the current density as a function of time given by the following expression:

$$\mathbf{J} = \begin{cases} \hat{0}; & t \leq 0 \\ \left(\frac{J_e t}{T}\right) \hat{k}; & 0 \leq t \leq T \\ J_e \hat{k}; & T \leq t \end{cases}$$

where $J_e$ is positive constant with units of amps per square meter.

a) Find the direction and magnitude of the magnetic field for the interval $0 \leq t \leq T$ in the regions: (i) $0 \leq r \leq a$; (ii) $r \geq a$, where $r$ is the distance from the symmetry axis of the wire.

Answer:

$0 \leq r \leq a$: We take an Amperean loop which is a circle of radius $r < a$. We have

$$\oint_{\text{closed path}} \mathbf{B} \cdot d\mathbf{s} = 2\pi r \mathbf{B}_\theta = \mu_0 \int \mathbf{J} \cdot \mathbf{n} \, da = \mu_0 \int_0^r 2\pi r' \, dr' \left(\frac{J t}{T}\right) = \mu_0 \pi r^2 \left(\frac{J t}{T}\right) \Rightarrow \mathbf{B} = \frac{r}{2} \mu_0 \left(\frac{J_e t}{T}\right) \hat{\theta}$$

$\hat{\theta}$ is counterclockwise looking from left

$r > a$: We take an Amperean loop which is a circle of radius $r > a$. We have

$$\oint_{\text{closed path}} \mathbf{B} \cdot d\mathbf{s} = 2\pi r \mathbf{B}_\theta = \mu_0 \int \mathbf{J} \cdot \mathbf{n} \, da = \mu_0 \int_0^a 2\pi r' \, dr' \left(\frac{J t}{T}\right) = \mu_0 \pi a^2 \left(\frac{J t}{T}\right) \Rightarrow \mathbf{B} = \frac{a^2}{2r} \mu_0 \left(\frac{J_e t}{T}\right) \hat{\theta}$$

$\hat{\theta}$ is counterclockwise looking from left
b) Suppose a square conducting loop with resistance $R$, and side $s$ is placed in the region $r > a$, such that the nearest side of the loop to the wire is a distance $b$ from the axis as shown in the end view figure below. What is the induced current in the square loop for the time interval $0 \leq t \leq T$? Draw the direction of the induced current on the figure.

![End view figure of a square loop](image)

**Answer:**

The direction of the current is clockwise as viewed from above. In the time interval $0 \leq t \leq T$, we have

$$\frac{d\Phi}{dt} = \left| \int_{\text{open surface } S} \mathbf{B} \cdot \hat{n} \, da \right| = \left| \int_{b}^{b+s} dr \int_{r}^{\frac{b+s}{r}} a^2 \mu_0 \left( \frac{J_e t}{T} \right) \right|$$

$$= \frac{sa^2}{2} \mu_0 \left( \frac{J_e t}{T} \right) \ln \left( \frac{b+s}{b} \right)$$

$$|I| = \frac{1}{R} \left| \frac{d\Phi}{dt} \right| = \mu_0 \frac{sa^2 J_e}{2RT} \ln \left( \frac{b+s}{b} \right)$$

**c)** What is the direction and magnitude of the force due to the induced current on the square loop during the time interval $0 \leq t \leq T$? What is the direction and magnitude of the torque due to the induced current on the square loop during the time interval $0 \leq t \leq T$?

**Answer:**

The force on the far side and the near side of the loop are equal and opposite, and cancel. The force on the left side of the loop is to the right and the force on the right side is to the left, giving a net force pushing the loop away from the wire. If $\hat{x}$ is to the right, then

$$\mathbf{F} = sI \left[ B_{r=b+s} - B_{r=b} \right] \hat{x} = sI \left( \frac{a^2}{2} \mu_0 \left( \frac{J_e t}{T} \right) \right) \left( \frac{1}{b} - \frac{1}{b+s} \right) \hat{x}$$

where
\[ |I| = \mu_0 \frac{sa^2J}{2RT} \frac{\varepsilon}{\varepsilon_1} \ln \left( \frac{b+s}{b} \right). \]
Problem 8 Alternating-Current Generator

An $N$-turn rectangular loop of length $a$ and width $b$ is rotated at a frequency $f$ in a uniform magnetic field $\mathbf{B}$ which points into the page, as shown in the figure below. At time $t = 0$, the loop is vertical as shown in the sketch, and it rotates counterclockwise when viewed along the axis of rotation from the left.

(a) Make a sketch depicting this “generator” as viewed from the left along the axis of rotation at a time $\Delta t$ shortly after $t = 0$, when it has rotated an angle $\theta$ from the vertical. Show clearly the vector $\mathbf{B}$, the plane of the loop, and the direction of the induced current.

**Solution:**

(b) Write an expression for the magnetic flux $\Phi$ passing through the loop as a function of time for the given parameters.

**Solution:** The dot product between the magnetic field and the unit normal is

$$\mathbf{B} \cdot \mathbf{n} = B \cos \theta(t)$$  \hspace{1cm} (0.1)

The angle $\theta(t) = \omega t + \theta_0$ where $\omega$ is the angular frequency is related to the frequency $f$ by $\omega = 2\pi f$, and $\theta_0 = 0$ is the angle between $\mathbf{n}$ and $\mathbf{B}$ and $t = 0$. The magnetic flux through the loop is

$$\Phi_{\text{magnetic}} = N \int_{\text{oneturn}} \mathbf{B} \cdot d\mathbf{A} = N \int_{\text{oneturn}} B \cos(2\pi ft) dA = NBab \cos(2\pi ft)$$  \hspace{1cm} (0.2)
(c) Show that an induced emf $\mathcal{E}$ appears in the loop, given by

$$\mathcal{E} = 2\pi fNBab \sin(2\pi ft) = \mathcal{E}_0 \sin(2\pi ft)$$

**Solution:** The time derivative of the magnetic flux is

$$\frac{d\Phi_{\text{magnetic}}}{dt} = NBab \frac{d}{dt} \cos(2\pi ft) = -2\pi fNBab \sin(2\pi ft) \quad (0.3)$$

So there is a non-zero electromotive force in the wire loop.

$$\mathcal{E} = -\frac{d}{dt} \int_{\text{open surface}} \mathbf{B} \cdot d\mathbf{A} = 2\pi fNBabs \sin(2\pi ft) = \mathcal{E}_0 \sin(2\pi ft) \quad (0.4)$$

where $\mathcal{E}_0 = 2\pi fNBab$. 
Problem 9 Bar Moving across a Current Loop

A conducting rod with zero resistance and length \( w \) slides without friction with velocity \( \vec{v} \) on two parallel perfectly conducting wires. Resistors \( R_1 \) and \( R_2 \) are connected across the ends of the wires to form a circuit, as shown. A constant magnetic field \( \vec{B} \) is directed out of the page. In computing magnetic flux through any surface, take the surface normal to be out of the page, parallel to \( \vec{B} \).

a) The magnetic flux (out of the plane of the figure) in the right loop of the circuit shown is (i) decreasing, (ii) increasing, or (iii) constant?
b) What is the magnitude of the rate of change of the magnetic flux through the right loop?
c) What is the current flowing through the resistor \( R_2 \) in the right hand loop of the circuit shown? Gives its magnitude and indicate its direction on the figure.
d) The magnetic flux in the left loop of the circuit shown is (i) decreasing, (ii) increasing, or (iii) constant?
e) What is the magnitude of the rate of change of the magnetic flux through the right loop?
f) What is the current flowing through the resistor \( R_1 \) in the left hand loop of the circuit shown? Gives its magnitude and indicate its direction on the figure.
g) What is the magnitude and direction of the magnetic force exerted on this rod?
a) The magnetic flux (out of the plane of the figure) in the right loop of the circuit shown is (i) decreasing, (ii) increasing, or (iii) constant?

**Answer:** Increasing.

b) What is the magnitude of the rate of change of the magnetic flux through the right loop?

**Answer:**

\[
\frac{d\Phi(t)}{dt} = \frac{d}{dt} BA = B \frac{d}{dt} A = BwV
\]

c) What is the current flowing through the resistor \( R_2 \) in the right hand loop of the circuit shown? Gives its magnitude and indicate its direction on the figure.

**Answer:**

The flux out of the page is increasing so the current is clockwise to make a flux into the page. We can determine the magnitude of the induced current from Faraday’ Law:

\[
I = \frac{|\mathcal{E}|}{R_2} = \frac{1}{R_2} \frac{d\Phi(t)}{dt} = \frac{BwV}{R_2}
\]
d) The magnetic flux in the left loop of the circuit shown is (i) decreasing, (ii) increasing, or (iii) constant?

**Answer:** Decreasing.

e) What is the magnitude of the rate of change of the magnetic flux through the right loop?

**Answer:**

\[
\frac{d\Phi(t)}{dt} = \frac{d}{dt} BA = B \frac{d}{dt} A = -BwV
\]

“Magnitude” is ambiguous – either a positive or negative number will do here. I use the negative sign to indicate that the flux is decreasing.

g) What is the current flowing through the resistor \( R_1 \) in the left hand loop of the circuit shown? Gives its magnitude and indicate its direction on the figure.

**Answer:**

The flux out of the page is decreasing so the current is counterclockwise to make a flux out of the page to make up for the loss. We can determine the magnitude of the induced current from Faraday’s Law:

\[
I = \frac{|\varepsilon|}{R_1} = \frac{1}{R_1} \frac{d\Phi(t)}{dt} = \frac{BwV}{R_1}
\]

h) What is the magnitude and direction of the magnetic force exerted on this rod?

**Answer:**

The total current through the rod is the sum of the two currents (they both go up through the rod). Using the right hand rule on \( \vec{F} = \vec{I} \times \vec{B} \) we see the force is to the right. You could also get this directly from Lenz. The magnitude of the force is:

\[
F = |\vec{F}| = ILB = \left( BwV \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right) wB = B^2w^2V \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]
Problem 10: Coupled Rods

Two rods 1 and 2 slide without friction on two conducting rails separated by a distance \( w \), in the presence of a constant background magnetic field \( \vec{B} \) (pointing out of the plane of the figure). The mass of each rod is \( m \) and each rod has a resistance \( R \). The rails have no resistance. At time \( t \), the rod on the left is a distance \( x_1(t) \) from the origin, and the rod on the right is a distance \( x_2(t) \) from the origin. The respective x-component of the velocities of the rods are given by \( v_1(t) = \frac{dx_1(t)}{dt} \) and \( v_2(t) = \frac{dx_2(t)}{dt} \). At \( t = 0 \) rod 2 (on the right) is moving to the right with speed \( v_{2,0} \), and rod 1 (on the left) is at rest. We want to find the subsequent x-component of the velocities of the rods for \( t > 0 \).

(a) At time \( t \), what is the magnetic flux through the circuit consisting of the two rods and the rails connecting them?

(b) At time \( t \), what is the current through the circuit consisting of the two rods and the rails connecting them. Give its magnitude and also give its sense (clockwise or counterclockwise) using a Lenz’s Law argument. You can make this argument at \( t = 0 \) if you wish, when \( v_1(t) \big|_{t=0} = 0 \) and \( v_2(t) \big|_{t=0} = v_{2,0} \).

(c) At time \( t \), what is the magnetic force \( \vec{F}_2 \) on rod 2 (on the right), and what is the magnetic force \( \vec{F}_1 \) on rod 1 (on the left)?

(d) Using Newton’s Second Law for the x-component of the accelerations \( F_2 = m \frac{dv_2(t)}{dt} \) and \( F_1 = m \frac{dv_1(t)}{dt} \), and the results of (c), derive equations for \( dv_1(t) / dt \) and \( dv_2(t) / dt \). Using these equations, derive differential equations for the two new functions defined by \( g(t) = v_1(t) + v_2(t) \) and \( h(t) = v_2(t) - v_1(t) \).
(e) What are the solutions for \( g(t) = v_1(t) + v_2(t) \) and \( h(t) = v_2(t) - v_1(t) \) given the initial conditions? Remember that \( v_1(t) \big|_{t=0} = 0 \) and \( v_2(t) \big|_{t=0} = v_{2,0} \). (Note: if the function \( D(t) \) satisfies the equation \( \frac{d}{dt} D(t) = -\frac{D(t)}{\tau} \), then \( D(t) = D_0 e^{-t/\tau} \); if the function \( E(t) \) satisfies the equation \( \frac{d}{dt} E(t) = 0 \), then \( E(t) = E_0 \).

(f) Solve for the x-component of the velocities of rods 1 and 2 at time \( t \) using the time dependence of \( g(t) \) and \( h(t) \). After a very long time, what are the speeds of rods 1 and 2?

Solution:

(a) At time \( t \), what is the magnetic flux through the circuit consisting of the two rods and the rails connecting them?

Answer:

\[ \Phi(t) = wB(x_2(t) - x_1(t)) \]

(b) At time \( t \), what is the current through the circuit consisting of the two rods and the rails connecting them. Give its magnitude and also give its sense (clockwise or counterclockwise) using a Lenz’s Law argument. You can make this argument at \( t = 0 \) if you wish, when \( v_1(t) \big|_{t=0} = 0 \) and \( v_2(t) \big|_{t=0} = v_{2,0} \).

Answer:

The EMF is given by:

\[ \varepsilon = -\frac{d\Phi}{dt} = -\frac{d}{dt} wB\left(x_2(t) - x_1(t)\right) = -wB\left(v_2(t) - v_1(t)\right) \]

The minus sign means that the EMF (and the direction of driven current) is clockwise. We could also get this from Lenz’s Law, as a clockwise current will produce a self-field into the page inside the rails, trying to offset the increase in flux due to the increasing area. The current is thus clockwise and the magnitude is given by:

\[ I = \frac{|\varepsilon|}{2R} = \frac{wB\left(V_2(t) - V_1(t)\right)}{2R} \]
(c) At time \( t \), what is the magnetic force \( \vec{F}_2 \) on rod 2 (on the right), and what is the magnetic force \( \vec{F}_1 \) on rod 1 (on the left)?

**Answer:**

The magnitude of the magnetic force is \( wIB \) for both rods, to the left on the right rod and to the right on the left rod (trying to bring them together). So:

\[
\vec{F}_{left} = -\vec{F}_{right} = \left(\frac{wB}{2R}\right)^2 \left(V_2(t) - V_1(t)\right) \hat{i}
\]

(d) Using Newton’s Second Law for the x-component of the accelerations \( F_2 = m \frac{dv_2(t)}{dt} \) and \( F_1 = m \frac{dv_1(t)}{dt} \), and the results of (c), derive equations for \( \frac{dv_1(t)}{dt} \) and \( \frac{dv_2(t)}{dt} \). Using these equations, derive differential equations for the two new functions defined by \( g(t) = v_1(t) + v_2(t) \) and \( h(t) = v_2(t) - v_1(t) \).

**Answer:**

\[
\frac{dv_1(t)}{dt} = \frac{(wB)^2(v_2(t) - v_1(t))}{2mR}
\]

and

\[
\frac{dv_2(t)}{dt} = -\frac{(wB)^2(v_2(t) - v_1(t))}{2mR}
\]

Adding these yields:

\[
\frac{dg(t)}{dt} = \frac{dv_1(t)}{dt} + \frac{dv_2(t)}{dt} = 0.
\]

Subtracting them yields:

\[
\frac{dh(t)}{dt} = \frac{dv_2(t)}{dt} - \frac{dv_1(t)}{dt} = -\frac{(wB)^2(v_2(t) - v_1(t))}{mR} = -\frac{(wB)^2}{mR} h(t)
\]

Thus

\[
\frac{dh(t)}{dt} = -\frac{(wB)^2}{mR} h(t) = -\frac{h(t)}{\tau}
\]
where the constant $\tau = \frac{mR}{(wB)^2}$.

(e) What are the solutions for $g(t) = v_1(t) + v_2(t)$ and $h(t) = v_2(t) - v_1(t)$ given the initial conditions? Remember that $v_1(t)|_{t=0} = 0$ and $v_2(t)|_{t=0} = v_{2,0}$. (Note: if the function $D(t)$ satisfies the equation $\frac{d}{dt}D(t) = -\frac{D(t)}{\tau}$, then $D(t) = D_0 e^{-t/\tau}$; if the function $E(t)$ satisfies the equation $\frac{d}{dt}E(t) = 0$, then $E(t) = E_0$).

**Answer:**

From the information given and the results of the previous part we have that $g(t) = g_0$ and $h(t) = h_0 e^{-t/\tau}$. Given our initial conditions

$$g(t) = v_{2,0} \quad \text{and} \quad h(t) = v_{2,0} e^{-t/\tau}$$

(f) Solve for the x-component of the velocities of rods 1 and 2 at time $t$ using the time dependence of $g(t)$ and $h(t)$. After a very long time, what are the speeds of rods 1 and 2?

**Answer:**

$$v_1(t) = \frac{1}{2} (g(t) - h(t)) = \frac{v_{2,0}}{2} (1 - e^{-t/\tau})$$

and

$$v_2(t) = \frac{1}{2} (g(t) + h(t)) = \frac{v_{2,0}}{2} (1 + e^{-t/\tau})$$

After a long time the exponentials decay to zero and both rods move at speed $v_{2,0} / 2$ to the right.
At the instant shown in the figure above, an electron at the position shown with charge $q = -e$ is moving with velocity $\vec{v} = v\hat{j}$. A very long wire carries a current in the direction shown in the figure. At the instant depicted, the force on the electron is

1. zero.
2. in the $+\hat{i}$-direction.
3. in the $-\hat{i}$-direction.
4. in the $+\hat{j}$-direction.
5. in the $-\hat{j}$-direction.
6. in the $+\hat{k}$-direction.
7. in the $-\hat{k}$-direction.

Answer ________________________.
**Answer 2.** Only the horizontal leg contributes to the magnetic field at the location of the moving charge. By the right hand rule the field is in the negative \( \hat{k} \)-direction, \( \vec{B} = -B \hat{k} \). Therefore the force on the charged particle is
\[
\vec{F}_q = -ev\hat{j} \times (-B \hat{k}) = evB \hat{i},
\]
which is in the \(+\hat{i}\)-direction.
Concept Questions 6 Biot-Savart

Consider the following loop of wire carrying a current $I$ as shown in the figure below.

Which of the following statements is true for the magnetic field at the point $P$ located at the origin?

1. The magnetic field at the point $P$ is zero.
2. The magnetic field at point $P$ is directed in positive $y$-direction.
3. The magnetic field at point $P$ is directed in negative $y$-direction.
4. The magnetic field at point $P$ is directed in positive $x$-direction.
5. The magnetic field at point $P$ is directed in negative $x$-direction.
6. The magnetic field at point $P$ is directed into the plane of the figure.
7. The magnetic field at point $P$ is directed out of the plane of the figure.

Answer: _________________________
Answer 6. The two legs along the horizontal axis do not contribute to the magnetic field at the point $P$. Outer semi-circular ring makes a smaller contribution to the magnitude of the magnetic field than the inner semi-circular ring. Field due to inner ring points into the plane of the figure. Field due to outer ring points out of the plane of the figure. Therefore field at center points into plane of figure.
Concept Question 8 Ampere’s Law

The figure shows three wires carrying currents $I_1$, $I_2$ and $I_3$, with an Ampèreian loop drawn around $I_1$ and $I_2$. The wires are all perpendicular to the plane of the paper. Which currents produce the magnetic field at the point $P$ shown in the sketch (circle one)?

1. $I_3$ only.
2. $I_1$ and $I_2$.
3. $I_1$, $I_2$ and $I_3$.
4. None of them.
5. It depends on the size and shape of the Ampèreian Loop.

Answer: _______________________
Answer 3. All three currents contribute to the magnetic field at every point in space. Ampere’s Law \(\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{enc}\), about the Amperian loop shown in the figure, requires that the line integral is equal to \(\pm \mu_0 (I_1 - I_2)\) depending on which circulation direction is used for the line integral. But the line integral is not equal to \((B)(\text{length of path})\) because \(\mathbf{B} \cdot d\mathbf{s}\) varies along the path, i.e. the direction of the magnetic field is not tangential along the path, and the magnitude of the magnetic field also varies along the path.
Concept Question 9 Ranking Circular Orbits

Two charged particles of identical charge move in circular orbits in the same uniform magnetic field that is perpendicular to the orbital plane. Particle A has twice the speed of particle B, \( v_A = 2v_B \). Particle A has one quarter the mass of the particle B, \( m_A = m_B / 4 \).

1. The radius of the orbit of particle B is four times the radius of the orbit of particle A.
2. The radius of the orbit of particle B is twice the radius of the orbit of particle A.
3. The radius of the orbit of particle B is equal to the radius of the orbit of particle A.
4. The radius of the orbit of particle B is one half the radius of the orbit of particle A.
5. The radius of the orbit of particle B is one quarter the radius of the orbit of particle A.

Answer: _________________________
**Answer 4.** By Newton’s Second Law the radius of the orbit is

\[ qvB = \frac{mv^2}{R} \Rightarrow R = \frac{mv}{qB}. \]

Because the charge and the field are the same, the ratio of the radii for the two particles is given

\[ \frac{R_B}{R_A} = \frac{m_B v_B}{m_A v_A} = \frac{4 m_B v_B}{m_B 2v_B} = 2. \]
Concept Question 19 Mutual Inductance

Consider the mutual inductance between two co-axial solenoids. The first is a very long outer solenoid with a certain number of turns and a fixed length and radius (see sketch). Inside that long solenoid, at its center, is a short inner solenoid with a certain number of turns, a radius much smaller than the radius of the outer solenoid, and a length which is much smaller than the length of the outer solenoid. When we double the number of turns of both solenoids, keeping their lengths the same, and half the radius of the inner solenoid, the mutual inductance of this system:

1. Goes up by a factor of 16.
2. Goes up by a factor of 8.
4. Goes up by a factor of 2.
5. Does not change.
8. Goes down by a factor of 8.
9. Goes down by a factor of 16.

**Answer:** _________________________
Answer 5.

Let $I_1$ be the current in the outer solenoid with $N_1$ turns and length $h_1$. The magnetic field from the outer solenoid is then $B_1 = \mu_0 (N_1 / h_1) I_1$. The inner solenoid has $N_2$ turns and radius $R_2$. Then the magnetic flux through inner solenoid due to field of outer solenoid is

$$\Phi_{12} = N_2 B_1 A_2 = N_2 \mu_0 (N_1 / h_1) I_1 \pi R_2^2.$$

The mutual inductance is then

$$M_{12} = \frac{\Phi_{12}}{I_1} = N_2 \mu_0 (N_1 / h_1) \pi R_2^2.$$

When we double the number of turns of both solenoids, keeping their lengths the same, and half the radius of the inner solenoid, the mutual inductance of this system does not change.
A positively charged particle of mass $m$ with charge $q > 0$ and speed $v$ enters a region between screens $S_1$ and $S_2$ that are separated by a distance $d$, where there are uniform electric and magnetic fields (see figure). The particle enters halfway between the plates. The electric field is created by two oppositely uniformly charged plates that are separated by a distance $h$. You may assume that the electric and magnetic fields are uniform between the screens (neglect edge effects). The magnitude of the electric field, $E$, and the magnitude of the magnetic field, $B$, are related by the expression $E = vB/2$. The distance between the charged plates satisfies the inequality $h > qBd^2/2mv$.

1. The charged particle will pass through the hole on screen $S_2$.

2. The charged particle will hit the upper charged plate.

3. The charged particle will hit the lower charged plate.

4. The charged particle will hit the screen $S_2$ above the hole.

5. The charged particle will hit the screen $S_2$ below the hole.

6. Not enough information is given to determine the trajectory of the particle.

**Answer:** _________________________
Answer 4. The force on the charge is given by

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = -qE\hat{j} + qv\hat{i} \times (-B\hat{k}) = -qE\hat{j} + qvB\hat{j} \]

Using the fact that \( E = \frac{vB}{2} \), the magnetic force on the charged particle is

\[ \mathbf{F} = \left(\frac{qvB}{2}\right)\hat{j}. \]

Newton’s Second Law is then \( \left(\frac{qvB}{2}\right)\hat{j} = m\mathbf{a} \), hence the acceleration is constant and pointing in the positive \( \hat{j} \)-direction with \( y \)-component given by \( a_y = \frac{qvB}{2m} \). The particle undergoes parabolic motion. The equations of motion are

\[
y(t) = \frac{1}{2} a_y t^2 = \frac{qvB}{4m} t^2.
\]

\[
x(t) = vt.
\]

When \( x(t_1) = vt_1 = d \Rightarrow t_1 = d/v \). Then \( y(t_1) = \frac{qvB}{4m} t_1^2 = \frac{qvBd^2}{4mv} < h \), because \( h > \frac{qBd^2}{2mv} \). Therefore the particle hits the screen \( S_2 \) above the hole.
Concept Question 22 Helmholtz Coils and Torque

In the Magnetic Force experiment you pushed an equal current through two coils in either a Helmholtz or Anti-Helmholtz configuration. You also had a magnetic dipole on a spring free to move along the central axis of the configuration.

Consider the two iron filings diagrams above, (A) and (B), which represent the magnetic fields generated by the coils in the Helmholtz and Anti-Helmholtz configurations in some order (you should be able to determine which is which). In both cases you adjust the spring so that the dipole is slightly above center. After letting the system settle you turn off the current. Identify which of the two diagrams represents the Helmholtz configuration, and in which of the situations (or neither or both), the dipole will move downwards when the current is turned off.

1. (A) is the Helmholtz configuration; in (A) the dipole will relax downward.
2. (A) is the Helmholtz configuration; in (B) the dipole will relax downwards.
3. (A) is the Helmholtz configuration; in neither case will the dipole relax downwards.
4. (A) is the Helmholtz configuration; in both cases the dipole will relax downwards.
5. (B) is the Helmholtz configuration; in (A) the dipole will relax downwards.
6. (B) is the Helmholtz configuration; in (B) the dipole will relax downwards.
7. (B) is the Helmholtz configuration; in neither case will the dipole relax downwards.
8. (B) is the Helmholtz configuration; in both cases the dipole will relax downwards.

Answer: _________________________
**Answer 5.** The configuration (a) is anti-Helmholtz and configuration (B) is Helmholtz. When the dipole is place slightly above the center in the anti-Helmholtz configuration it will feel a torque that will align the dipole moment in the positive z-direction and then it will feel a force upwards. When the current is turned off it will relax downward. When the dipole is place slightly above the center in the Helmholtz configuration it will feel a torque that will align the dipole moment in the positive z-direction and it will no force (or a negligibly small force downwards). When the current is turned off it will not move up or down (or relax a very small distance upwards.)
Concept Question 23 Ranking Resistance

Two resistors are made out of the same material, but have different dimensions, as shown in the figures. The current through these two resistors is in the direction shown. If the resistor on the left has a resistance of $1 \Omega$, the resistor on the right will have a resistance of

1. 4 ohm
2. 8 ohm
3. 16 ohm
4. 32 ohm
5. 2 ohm
6. 1 ohm
7. 0.5 ohm
8. None of the above.

Answer: _________________________
**Answer 7.** The resistance for the resistor on the left is \( R_L = \rho_r \frac{2h}{(4\pi a^2)} = \frac{\rho_r h}{(2\pi a^2)} \).

The resistance for the resistor on the right is \( R_R = \frac{\rho_r h}{\pi a^2} = \frac{1}{2} R_L = 0.5 \Omega \).
Test Problem 4 Faraday’s Law and Force

A rectangular loop of wire with resistor $R$ is placed next to a straight wire carrying a current that is varying in time according to

$$I(t) = I_0 - at \ ; \ 0 < t < I_0 / a$$

During the time interval $0 < t < I_0 / a$, the rectangular loop will feel

1. a net force to the right.
2. a net force to the left.
3. a net force down, toward the wire.
4. a net force up, away from the wire.
5. a net force into the page.
6. a net force out of the page.
7. no net force.

Answer: _________________________
**Answer 3.** The magnetic field from the long wire is directed into the plane of the figure and decreasing. Therefore an induced current flow is clockwise so bottom leg has current to the left and a force toward wire, which is larger (it’s closer to wire) than force away from wire on upper branch.
Test Problem 3 Induced Electric Field

The figure on the left shows a side view of a solenoid with a current $I$. On the right is a top view with electric and magnetic fields $\vec{E}$ and $\vec{B}$ at time $t$. At this time

1. the current $I$ is increasing in time
2. the current $I$ is constant
3. the current $I$ is decreasing in time

Answer: _________________________
Answer 1.

The current is increasing in time. Suppose you placed a circular conducting ring exactly where the electric field is depicted in the figure. Then an induced current would be directed in the same clockwise direction as the electric field as seen from above. This induced current would produce magnetic flux out into the plane of the top view figure (above right). Because the induced flux is opposed to the change of the flux due to the magnetic field in the solenoid, this implies that the magnetic field of the solenoid is increasing in time. Therefore the current in the solenoid is also increasing in time.

Alternatively, choose a unit normal for the flux pointing in the same direction as the magnetic field of the solenoid. This implies that the circulation direction is counterclockwise in the top view figure (above right). Because the electric field is clockwise as in that figure, the emf generated by the line integral of the electric field is negative. This negative emf is equal to the negative change of the magnetic flux which implies that the change of magnetic flux is positive, indicating that the magnetic field is increasing in time. Therefore the current in the solenoid is also increasing in time in agreement with our other argument.