Blackbody Radiation

As derived on Page 312 of B&B, the energy density of electromagnetic waves in thermal equilibrium with their surroundings is given by

\[ U(T) = a T^4, \]

where

\[ a = \frac{8\pi^5 k^4}{15h^3 c^3} = \frac{\pi^2 k^4}{15h^3 c^3} \]

and the substitution \( \hbar \equiv h/2\pi \) has been made. These waves will be a source of radiation. To find the intensity of this radiation (power per unit area), consider a boundary between two regions, one at temperature \( T_1 \) and the other at \( T_2 \). If a small hole is made in this boundary, energy will flow from the higher temperature region to the lower (if \( T_1 = T_2 \), the regions are in equilibrium and there is no net energy flow either way). In this case, “small” means that the presence of the opening does not affect either \( T_1 \) or \( T_2 \), and that the radiation from one region to the other depends only on the temperature of the region where the radiation originates.

So, we will consider a region of temperature \( T \) separated from a vacuum by a boundary, with a small hole of area \( A \). We will need to make two further assumptions; first, that the energy is associated with electromagnetic waves in the absence of charges, and hence is propagated at the speed of light \( c \), and second that the radiation is isotropic (\( iso = \text{same, trop = direction} \)), meaning that the radiation intensity is the same in all directions.

So, then, any radiation that leaves the opening at a given time was at a distance \( r = ct \) a time \( t \) previously. This set of points is a hemisphere, shown in cross-section.
The energy leaving the opening in time \( dt \) must have come from a shell of thickness \( c \, dt \). However, not all of this energy in the shell will leave through the opening. Using the isotropic property of the radiation, consider a volume \( dV \), the usual volume element in spherical polar coordinates with origin at the opening and azimuth (\( z \)-axis) perpendicular to the boundary.

\[
dV = (dr)(r \, d\theta)(r \sin \theta \, d\phi)
\]

The electromagnetic energy in this volume is \( dE = U \, dV \) (here, \( "E" \) is energy, not electric field amplitude). This radiation will radiate isotropically; the fraction that leaves through the opening will be the ratio of the solid angle subtended by the opening \( as \ seen \ from \ the \ point \ where \ the \ volume \ element \ dV \ is \ to \ the \ total \ solid \ angle. \) This ratio is \( A \cos \theta/4\pi r^2 \). Thus,

\[
\frac{A \cos \theta}{4\pi r^2} \, dE = A \, dS \, dt, \quad \text{or} \quad \frac{\cos \theta}{4\pi r^2} \, dV \, U = dS \, dt.
\]

Using

\[
dV = r^2 \sin \theta \, dr \, d\theta \, d\phi = r^2 \sin \theta \, d\theta \, d\phi \, c \, dt,
\]

\[
dS = \frac{Uc}{4\pi} \cos \theta \sin \theta \, d\theta \, d\phi \quad \text{and}
\]

\[
S = \int dS = \frac{Uc}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta \\
= \frac{Uc}{2} \int_0^{\pi/2} \sin \theta \, d(\sin \theta) \\
= \frac{Uc}{2} \cdot \frac{1}{2} = \frac{Uc}{4}.
\]

(Note the range of integration on \( \theta \), corresponding to half a sphere). Thus,

\[
S = \frac{Uc}{4} = \frac{c}{4} a T^4 = \sigma T^4, \quad \text{where}
\]

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\[ \sigma = \frac{c}{4} a = \frac{\pi^2 k^4}{60h^3 c^2} = 5.67051 \times 10^{-8} \frac{W}{m^2 \cdot K^4} \]

is the “Stefan-Boltzmann” constant.

Thus, the net rate of heat flow across an aperture of area \( A \) between two regions of radiation at temperatures \( T_1 \) and \( T_2 \) is

\[ H = \sigma A (T_1^4 - T_2^4). \]

Note; You need the result

\[ \frac{S_1}{S_2} = \left( \frac{T_1}{T_2} \right)^4 \]

to do B&B problem 4.6. The numbers in that problem have been rigged! Calculators are not necessary!