Coupled Linear Oscillators

You need not memorize any of this, but please don’t toss it; we’ll be needing some of the results soon.

In these notes, matrices will be denoted by boldface capital letters.

For a system of coupled oscillators subject to linear restoring forces, we found that the equations of motion may be written in matrix form;

\[
\mathbf{M} \ddot{\mathbf{x}}(t) = \mathbf{K} \mathbf{x}(t),
\]

where \( \mathbf{x}(t) \) will be taken as a column vector, each of whose components is one of the dependent variables, \( \mathbf{M} \) is a diagonal matrix with positive elements, and \( \mathbf{K} \) is symmetric. For example, we have seen that for the simple system of two masses connected by springs, as shown,

![Diagram of two masses connected by springs](image)

the coupled equations

\[
\begin{align*}
m_1 \ddot{x}_1 &= -kx_1 - k'(x_1 - x_2) \\
m_2 \ddot{x}_2 &= -kx_2 - k'(x_2 - x_1)
\end{align*}
\]

became

\[
\begin{bmatrix}
m_1 & 0 \\
0 & m_2
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix} =
\begin{bmatrix}
-k - k' & k' \\
 k' & -k - k'
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

It will be helpful to use row vectors also; if

\[
\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \bar{\mathbf{x}}^T = [x_1, \ldots, x_n].
\]

The “\( T \)” superscript is for “transpose”. From the symmetry of \( \mathbf{M} \) and \( \mathbf{K} \), note that

\[
\bar{\mathbf{x}}^T \mathbf{M} = (\mathbf{M} \bar{\mathbf{x}})^T, \quad \bar{\mathbf{x}}^T \mathbf{K} = (\mathbf{K} \bar{\mathbf{x}})^T.
\]
Note also that $\vec{x}^T \vec{x}$ is a scalar; in fact, $\vec{x}^T \vec{x} = \vec{x} \cdot \vec{x}$.

So, let’s look for solutions to (1) of the form

$$\ddot{\vec{x}} = -\omega^2 \vec{x}. \quad (2)$$

Note that, if $\vec{x}_0$ is a constant vector, $\vec{x} = \vec{x}_0 e^{i\omega t}$, $\vec{x} = \vec{x}_0 e^{-i\omega t}$, $\vec{x} = \vec{x}_0 \cos \omega t$, $\vec{x} = \vec{x}_0 \sin \omega t$ or any linear combination will suffice. Note that in this usage, $\vec{x}_0$ is not necessarily $\vec{x}(0)$. The specific functional form won’t matter, as long as (2) holds.

Substitution into (1) yields

$$(K + \omega^2 M) \vec{x} = \vec{0}. \quad (3)$$

The only way for non-trivial solutions of (3) to exist is to have $\det (K + \omega^2 M) = 0$. This allows us to solve algebraically for $\omega^2$, and hence for $\vec{x}_0$ (within constant multiples). Specifically, let $\vec{x}_{\alpha} = \vec{x}_{\alpha 0} e^{-i\omega_{\alpha} t}$. Now, then, what are meant by “normal modes”? Who are we to judge anyone else’s normalcy? This is, after all MIT. Well, consider

$$\vec{x}_{\beta 0}^T K \vec{x}_{\alpha 0} = \vec{x}_{\beta 0}^T (K \vec{x}_{\alpha 0}) = \vec{x}_{\beta 0}^T (-\omega_{\alpha}^2 M \vec{x}_{\alpha 0}) = -\omega_{\alpha}^2 (\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0}).$$

But,

$$\vec{x}_{\beta 0}^T K \vec{x}_{\alpha 0} = (\vec{x}_{\beta 0}^T K) \vec{x}_{\alpha 0} = (K \vec{x}_{\beta 0})^T \vec{x}_{\alpha 0}$$
$$= (-\omega_{\beta}^2 M \vec{x}_{\beta 0})^T \vec{x}_{\alpha 0} = -\omega_{\beta}^2 (M \vec{x}_{\beta 0})^T \vec{x}_{\alpha 0}$$
$$= -\omega_{\beta}^2 (\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0}).$$

So, we have

$$-\omega_{\alpha}^2 (\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0}) = -\omega_{\beta}^2 (\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0}), \quad \text{or}$$

$$(\omega_{\alpha}^2 - \omega_{\beta}^2) (\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0}) = 0,$$

which means that if $\omega_{\alpha}^2 \neq \omega_{\beta}^2$,

$$\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0} = 0,$$

and this is the interpretation of “normal modes” that we need.

If $\omega_{\alpha}^2 = \omega_{\beta}^2$ for some $\alpha \neq \beta$, the above relation still holds, if $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ are chosen properly. First, note that if $\omega_{\alpha}^2 = \omega_{\beta}^2$, (3) is satisfied for $\omega^2 = \omega_{\alpha}^2$ and $\vec{x} = a\vec{x}_{\alpha 0} + b\vec{x}_{\beta 0}$ for any scalars $a$ and $b$. Thus, $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ may be chosen such that $\vec{x}_{\beta 0}^T M \vec{x}_{\alpha 0} = 0$ if $\vec{x}_{\alpha 0}$ and $\vec{x}_{\beta 0}$ are linearly independent, and a result from linear
algebra shows that they will be. However, the physics shows why the vectors \( \vec{x}_{\alpha 0} \) and \( \vec{x}_{\beta 0} \) may always be chosen to be normal.

The specific values of \( \omega^2_\alpha \) and the components of \( \vec{x}_{\alpha 0} \) will depend on the values of the masses (i.e., the diagonal elements of \( \mathbf{M} \)). If any frequency appears as a multiple root of (3), changing a value of some mass slightly will remove the degeneracy, the \( \omega^2_\alpha \) will be different, and the modes will be necessarily normal. A small change in the elements of \( \mathbf{M} \) cannot change the normality of the vectors. Please note that this explanation does not constitute a proof; it relies on our belief, on physical grounds, that a small change in the linear system cannot grossly affect the normalcy criteria.

We saw in class that if the masses are all the same, \( \mathbf{M} \) is a scalar multiple of the identity matrix, and the normality condition reduces to \( \vec{x}_{\beta 0} \cdot \vec{x}_{\alpha 0} = 0 \). We also saw that if the masses are not identical, the modes are normal when “weighted” by the masses.

Before moving on, let’s take advantage of the physics to help us do some math; define the matrix \( \mathbf{M}^{\frac{1}{2}} \) as the diagonal matrix whose elements are the non-negative square roots of the masses, so that \( \mathbf{M}^{\frac{1}{2}} \mathbf{M}^{\frac{1}{2}} = \mathbf{M} \). Then, rewrite (1) as

\[
\mathbf{M} \ddot{\vec{x}} = \mathbf{M}^{\frac{1}{2}} \mathbf{M}^{\frac{1}{2}} \ddot{\vec{x}} = \mathbf{K} \vec{x} = \mathbf{K} \mathbf{M}^{-\frac{1}{2}} \mathbf{M}^{\frac{1}{2}} \vec{x} \quad , \quad \text{or}
\]

\[
\mathbf{M}^{\frac{1}{2}} \ddot{\vec{x}} = \left( \mathbf{M}^{-\frac{1}{2}} \mathbf{K} \mathbf{M}^{-\frac{1}{2}} \right) \mathbf{M}^{\frac{1}{2}} \vec{x},
\]

where \( \mathbf{M}^{-\frac{1}{2}} = \left( \mathbf{M}^{\frac{1}{2}} \right)^{-1} \). This last point may seem trivial, but fractional powers of matrices are not as easily defined or determined as for scalars; our form for \( \mathbf{M} \) makes it easy. Also, note that if the masses are not the same, \( \mathbf{M}^{-\frac{1}{2}} \mathbf{K} \neq \mathbf{K} \mathbf{M}^{-\frac{1}{2}} \). So, define new coordinates by \( \vec{y}(t) = \mathbf{M}^{\frac{1}{2}} \vec{x}(t) \). With \( \mathbf{K}' \equiv \mathbf{M}^{-\frac{1}{2}} \mathbf{K} \mathbf{M}^{-\frac{1}{2}} \), (1) becomes

\[
\ddot{\vec{y}} = \mathbf{K}' \vec{y}.
\]

Our conditions for normal modes are then

\[
\det \left( \mathbf{K}' + \omega^2 \mathbf{I} \right) = 0, \quad \vec{y}_{\alpha 0} \cdot \vec{y}_{\beta 0} = 0.
\]

Note that this is a mathematical convenience; if the \( x_1, \ldots, x_n \) represent lengths, changing to \( y_1, \ldots, y_n \) changes each length by a different factor, and that ain’t physics. We will want the mathematical convenience at a later date.

For our purposes, we will need real solutions, and it would be nice to know that real solutions exist; this would mean that \( \omega_\alpha \) is real for all \( \alpha \) (if \( \omega_\alpha \) is real, all
of the elements of $\bar{x}_{\alpha 0}$ may be chosen real). All we need is the fact that both $K$ and $M$ are real and symmetric. Then, if $K \bar{x}_{\alpha 0} = -\omega^2 M \bar{x}_{\alpha 0}$,

$$\bar{x}^T_{\alpha 0} K = (K \bar{x}_{\alpha 0})^T = -\omega^2 (M \bar{x}_{\alpha 0})^T$$

and,

$$(\bar{x}^*_{\alpha 0} K) = (K \bar{x}_{\alpha 0}^*)^T = (-\omega^2 M \bar{x}_{\alpha 0})^T = (-\omega^2)^* (\bar{x}_{\alpha 0}^* M).$$

Thus, we have

$$\bar{x}^*_{\alpha 0} (K \bar{x}_{\alpha 0}) = \bar{x}^*_{\alpha 0} (-\omega^2 M \bar{x}_{\alpha 0}) = -\omega^2 (\bar{x}_{\alpha 0}^* M \bar{x}_{\alpha 0}),$$
$$\bar{x}^*_{\alpha 0} (K) \bar{x}_{\alpha 0} = (-\omega^2)^* (\bar{x}_{\alpha 0}^* M) \bar{x}_{\alpha 0} = (-\omega^2)^* (\bar{x}_{\alpha 0}^* T M \bar{x}_{\alpha 0}),$$

so $(\omega^2)^* = \omega^2$, and $\omega^2$ is real. Those familiar with the terminology of linear algebra will note that we have introduced the Hermitian conjugates of the $\bar{x}_{\alpha 0}$, and that because both $K$ and $M$ are real and symmetric, they are Hermitian, as is $K'$.

Showing that $\omega^2$ is non-negative, so that $\omega$ is real, is best done by appeal to the physics. Specifically, $\bar{x} = \bar{0}$ must be a stable configuration point (or neutrally stable, if $\omega = 0$ for some $\alpha$). A way to see this is to consider the potential energy $V(x_1, \ldots, x_n)$ of the system due to the forces represented by $K$; each component of $K$ is then

$$K_{jk} = -\frac{\partial^2 V}{\partial x_j \partial x_k} |_{\bar{x} = \bar{0}}.$$  

Apart from the minus sign, $K$ is a Hessian matrix, a term you might have encountered in 18.02 or the equivalent. We’ll just use the calculus result, that $V$ is a minimum at $\bar{x} = \bar{0}$ if the eigenvalues of $K$ are negative. We could then use the result from linear algebra that if the eigenvalues of $K$ are negative, the eigenvalues of $K'$ (which are not the eigenvalues of $K$) are negative. Or, we can define $V'(y_1, \ldots, y_n) = V(\sqrt{m_1} y_1, \ldots, \sqrt{m_n} y_n)$, using $y_i = \frac{x_i}{\sqrt{m_i}}$, as in (4), and then observe that if $V$ is a minimum at $\bar{x} = \bar{0}$, $V'$ is a minimum at $\bar{y} = \bar{0}$.

So, now that we know that they exist, let’s look for real solutions (e.g., sines and cosines), specifically

$$\bar{x}(t) = \sum_{j=1}^{n} (a_j \bar{x}_{j0} \cos \omega_j t + b_j \bar{x}_{j0} \sin \omega_j t), \quad \bar{x}(0) = \bar{x}_0, \quad \dot{\bar{x}}(0) = \bar{v}_0,$$

where the $a_j$, $b_j$ are constants, and $\bar{x}_{j0}$ and $\omega_j$ are as found previously. Note that $\bar{x}_0$, without a $j$ or $\alpha$ subscript, is the initial value $\bar{x}(0)$, and is not, in general, equal to $\bar{x}_{j0}$ for any $j$. In terms of the vectors $\bar{x}_{j0}$,

$$\bar{x}_0 = \sum_{j=1}^{n} a_j \bar{x}_{j0}, \quad \bar{v}_0 = \sum_{j=1}^{n} \omega_j b_j \bar{x}_{j0}.$$

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To find the $a_j$ and $b_j$, use the linearity of $M$ and the normalcy condition;

$$M \ddot{\bar{x}}_0 = \sum_{j=1}^{n} a_j M \ddot{x}_j, \quad \ddot{x}_j M \ddot{\bar{x}}_0 = \sum_{j=1}^{n} a_j \ddot{x}_{j0}^T M \ddot{x}_{j0}. $$

The sum is of $n$ terms, all but one of which vanish. That term will be for $j = k$, so

$$\ddot{x}_{k0}^T M \ddot{\bar{x}}_0 = a_k \ddot{x}_{k0}^T M \ddot{x}_{k0}$$

and so

$$a_k = \frac{\ddot{x}_{k0}^T M \ddot{\bar{x}}_0}{\ddot{x}_{k0}^T M \ddot{x}_{k0}}.$$  

This is analogous to $A_x = (\hat{i} \cdot \vec{A}) / (\hat{i} \cdot \vec{i})$, where $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$. Similarly,

$$b_k = \frac{1}{\omega_k \ddot{x}_{k0}^T M \ddot{x}_{k0}}$$

and the solution to (5) is

$$\ddot{x}(t) = \sum_{j=1}^{n} \frac{\ddot{x}_{j0}}{\ddot{x}_{j0}^T M \ddot{x}_{j0}} \left( \ddot{x}_{j0}^T M \ddot{\bar{x}}_0 \cos \omega_j t + \frac{1}{\omega_j} \ddot{x}_{j0}^T M \ddot{\bar{v}}_0 \sin \omega_j t \right).$$

(The situation where $\omega_j = 0$ for some $j$ is not hard to incorporate, and is often made part of an assignment.) This may look like a mess, but consider; once we find the $\ddot{x}_{k0}$ and $\omega_k$, and compute $M \ddot{x}_0$ and $M \ddot{v}_0$, that’s it!

Well, what’s the point? Consider now a driven system,

$$M \dddot{x} = K \dddot{x} + \vec{F}(t),$$

where $\vec{F}$ is a column vector representing the external forces. Then, look for

$$\dddot{x}(t) = \sum_{k=1}^{n} g_k(t) \dddot{x}_{k0}, \quad \dddot{x} = \sum_{k=1}^{n} \dddot{g}_k \dddot{x}_{k0},$$

where the $\dddot{x}_{k0}$ are the vectors found previously. Then,

$$M \dddot{x} = \sum_{k=1}^{n} g_k(t) M \dddot{x}_{k0} = \sum_{k=1}^{n} g_k K \dddot{x}_{k0} + \vec{F}(t).$$

But, remember that $K \dddot{x}_{k0} = -\omega_k^2 M \dddot{x}_{k0}$, so

$$\sum_{k=1}^{n} (\dddot{g}_k + \omega_k^2 g_k) M \dddot{x}_{k0} = \vec{F}(t),$$

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and, as before,
\[ \sum_{k=1}^{n} (\ddot{g}_k + \omega_k^2 g_k) \vec{x}^T_{j0} \vec{M} \vec{x}_{k0} = \vec{x}^T_{j0} \vec{F} = \vec{x}_{j0} \cdot \vec{F}. \]

But the sum vanishes except for the \( j = k \) term; then,
\[ \ddot{g}_k + \omega_k^2 g_k = \frac{1}{\vec{x}^T_{k0} \vec{M} \vec{x}_{k0}} \vec{x}_{k0} \cdot \vec{F}(t). \]

The intial conditions for \( g_k \) are similarly found to be
\[ g_k(0) = \frac{\vec{x}^T_{k0} \vec{M} \vec{x}_0}{\vec{x}^T_{k0} \vec{M} \vec{x}_{k0}}, \quad \dot{g}_k(0) = \frac{\vec{x}^T_{k0} \vec{M} \vec{v}_0}{\vec{x}^T_{k0} \vec{M} \vec{x}_{k0}}. \]

This is a second order inhomogeneous equation. The solution to the homogeneous part is well known; the complete solution depends, of course, on \( \vec{F} \), and may be found by variation of parameters, undetermined coefficients, annihilators, Green’s functions, Laplace transforms or Tarot cards.

The main point here is; if any component of \( \vec{F} \) is sinusoidal with any frequency \( \omega_k \) (unless \( \vec{x}_{k0} \cdot \vec{F} \) vanishes), there will be one normal mode of the system that will be driven at resonance. This is when interesting things happen.