An After-the-Fact Discussion of Part of Problem 4.5
and
a Glimpse into the Future

Equations 4.16, 4.22 and 4.23, as well as Problem 4.4, all involve terms of the form
\[
\frac{\sin(\omega t - kr)}{r}.
\]
In Eq. 4.16, it was assumed that \( \Delta l \) is “short”, so that no integral was needed. What “short” meant was (a) \( \Delta l \ll r \), which has been sort of assumed from the beginning of Chapter 4, and (b) \( k\Delta l \ll 1 \), or \( \Delta l \ll \lambda \). The other uses still assume \( l \ll r \) (\( l \) as opposed to \( \Delta l \), now), but \( l \) might be comparable to \( \lambda \).

In any case, with \( |z| \leq l \), we have

\[
r^2 = R^2 + z^2 - 2Rz \cos \psi.
\]

Note that in Figure 4.11(b), \( \psi = \theta \), but in Problem 4.4(e), \( \psi = \phi \), and \( z \to x \), but in Problem 4.4(c), \( \psi = \frac{\pi}{2} \). We don’t want to be confused by angle labels, so I’ll stick to \( \psi \) as our generic angle.
Anyhow, the above form for \( r = r(R, z, \phi) \) is rarely useful without some approximations. For practical purposes, we will usually have \( l \ll R \), hence \( z \ll R \), and so

\[
r = R \left( 1 - \frac{2z}{R} \cos \psi + \frac{z^2}{R^2} \right)^{1/2}
\]

\[
\sim R \left( 1 - \frac{z}{R} \cos \psi + \frac{z^2}{2R^2} + \frac{z^2}{2R^2} \cos \psi \right)
\]

\[
= R \left( 1 - \frac{z}{R} \cos \psi + \frac{z^2}{2R^2} (1 + \cos \psi) \right)
\]

\[
= R \left( 1 - \frac{z}{R} \cos \psi + \left( \frac{z}{R} \cos \psi \right)^2 \right),
\]

keeping terms to order \((z/R)^2\).

Well, here’s the deal; for \( z \ll R \), we can ignore the second-order \((z/R)^2\) term, and \( r \sim R - z \cos \psi \). But what, you may ask, if \( \psi = \pm \frac{\pi}{2} \), so \( \cos \psi = 0 \)? Then, \( \cos^2 \left( \pm \frac{\psi}{2} \right) = \frac{1}{2} \), so \( r \sim R + \frac{z^2}{2R^2} \) (which we could have obtained directly from \( r^2 = R^2 + z^2 \)).

The question is; when \( \cos \psi = 0 \), why don’t we include the \((z/R)^2\) term? The answer is, because we don’t want to! This may seem like a cheap shot, but it’s really not. What’s happening is that including the \((z/R)^2\) correction is of the same order as accounting for the variation of \( r \) in the denominator of \( \sin(\omega t - kr)/r \). Yes, the correction is there, but we are usually justified in ignoring it.

Now, to be a real stinker, I introduce the reason for this explanation; in Chapter 8, Equations 8.64-8.67, we do include this correction, and it’s essential. In fact, it’s the only way Eq. 8.65 can be integrated (please note that here, \( R \to z \), but \( z \to \rho \) ). However, sneak a peek at Figure 8.22, and note that the oscillatory part is \( R \sim \lambda \) if \( \lambda \sim l \), so the previously used approximation is not valid.

Well, then, what happens if we’re off-axis in Figure 8.21? Then, we can’t do the integral unless \( z \gg a \). Then, things get interesting. Stay tuned.

A good reference for the criteria to determine if an observer is in the “local” (or “near”) field or the “far” (or “distant”) field, see Rojansky, Electromagnetic Fields and Waves, Chapter 24, but watch out for unorthodox notation.