Normal Modes for Continuous Systems

In class, we found solutions of the Wavy Quation
\[ \psi = \psi(x, t), \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \]  
for a stretched string of uniform mass per unit length \( \mu = m_s / L \), under tension \( T \), so that \( v^2 = T / \mu \).

For the case of a string fixed at \( x = 0 \) but with a massive ring (mass \( M_r \)) free to slide perpendicular to the \( x \)-direction at \( x = L \), the boundary conditions become
\[ \psi(x, 0) = 0, \quad M_r \frac{\partial^2 \psi}{\partial t^2}\bigg|_{x=L} = -T \frac{\partial \psi}{\partial x}\bigg|_{x=L}. \]  
When solutions of the form
\[ \psi(x, t) = X(x) T(t) = X(x) e^{i\omega t} \]
are sought (note the distinction between the tension \( T \) and the function \( T \)), (1) and (2) become, after using separation of variables and factoring out \( T(t) \),
\[ X'' + \frac{\omega^2}{v^2} X = 0, \quad X(0) = 0, \quad \frac{\omega^2}{v^2} X(L) = \frac{\mu}{M_r} X'(L). \]  
We can find explicit solutions to (3) in terms of the roots of a transcendental equation, and you have a problem investigating the orthogonality of those solutions by direct integration. These notes will show how the proper orthogonality relation may be demonstrated without finding explicit forms for \( X(x) \).

Let \( X_m(x) \) be a solution of (3) with \( \omega = \omega_m \), and \( X_n \) be a solution with \( \omega = \omega_n \). Then, consider the combination
\[ X_m'' X_n - X_m X_n'' = \frac{d}{dx} [X_m' X_n - X_m X_n']. \]  
(Fill in the missing step!) But, from (3), \( X_m'' = -\frac{\omega_m^2}{v^2} X_m \), \( X_n'' = -\frac{\omega_n^2}{v^2} X_n \), so
\[ (\omega_n^2 - \omega_m^2) X_m X_n = \frac{d}{dx} [X_m' X_n - X_m X_n']. \]
Integrating from 0 to \( L \),
\[ \frac{1}{v^2} (\omega_n^2 - \omega_m^2) \int_0^L X_m(x) X_n(x) \, dx = [X_m'(x) X_n(x) - X_m(x) X_n'(x)]_0^L \]
\[ = X_m'(L) X_n(L) - X_m(L) X_n'(L). \]
Using the boundary condition at $L$ (from (3)) to eliminate $X'_m$ and $X'_n$ in favor of $X_m$ and $X_n$,

$$
\frac{1}{\nu^2} (\omega_n^2 - \omega_m^2) \int_0^L X_m(x) X_n(x) \, dx = \frac{Mr}{\mu \nu^2} (\omega_m^2 - \omega_n^2) X_m(L) X_n(L).
$$

So, if $\omega_m^2 \neq \omega_n^2$,

$$
\int_0^L \mu X_m(x) X_n(x) \, dx + Mr X_m(L) X_n(L) = 0,
$$

and this is our orthogonality relation. The important physical point to note is that the integral represents “breaking the string into small pieces”, multiplying by the mass ($\mu \, dx$) of each piece, multiplying by the displacements of the independent modes at that point, and adding. The term $Mr \, X_m(L)X_n(L)$ corresponds to the finite mass at $x = L$.

The above is a specific application of the theory of Sturm-Liouville problems. For more on this subject, consult any decent Differential Equations text or ESG’s self-paced study units and notes, specifically Why we care about Bessel Functions, linked from the page at

http://web.mit.edu/18.03-esg/www/notes/TofC.html