Announcements

• There will be one more problem set in 8.322, due on May 16. I’ll post it as soon as possible

Reading topics for this period

• Formal scattering theory; Born Approximation; Separable Potential

Reading Recommendations 11


• Gottfried, Partial waves, §8.1; Formal development, 8.2(a); Born approximation, 8.3(a),(b).

• Schiff has a lengthy and pretty good section on scattering theory. §5 for basics and §9 for approximation methods.

Problem Set 11

Topics covered in the problems

• The phase shift and the radial wave function.

• Scattering from atoms in the first Born approximation.

• The Born approximation for a δ-shell

• Spin dependent scattering
Problems

1. A formula for the phase shift (postponed from last week.)

   In our studying of scattering theory we derived a general formula for elastic scattering amplitude from a potential $V(\vec{x})$,

   $$f(\vec{k}', \vec{k}) = \sum_\ell (2\ell + 1) f_\ell(E) P_\ell(\cos \theta_{\vec{k}'\vec{k}}) = -\frac{2m}{4\pi} (2\pi)^3 \langle \vec{k}' | V | \psi_+^{(+)\vec{k}} \rangle$$

   Here, $E = k^2/2m$, $f_\ell(E) = e^{i\delta_\ell(E)/k}$, the state $|\vec{k}\rangle$ is a plane wave, $\langle \vec{x}|\vec{k}\rangle = e^{i\vec{k} \cdot \vec{x}}$, and the state $|\psi_+^{(+)}\rangle$ is a solution regular at the origin and obeying outgoing wave boundary conditions. In lecture we showed that for a spherically symmetric potential,

   $$\langle \vec{x}|\psi_+^{(+)}\rangle = \sum_\ell (2\ell + 1) i^\ell e^{i\delta_\ell(k)} P_\ell(\cos \theta_{\vec{k}\vec{x}}) R_\ell^E(r),$$

   where $R_\ell$ is normalized so

   $$\lim_{r \to \infty} R_\ell(k, r) = \frac{1}{kr} \sin(kr - \ell\pi/2 + \delta_\ell(k)).$$

   (a) From this starting point, derive an exact expression for the phase shift in the $\ell$th partial wave,

   $$\sin \delta_\ell(k) = -2mk \int_0^\infty dr r^2 V(r) j_\ell(kr) R_\ell^E(k, r)$$

   Note that three different angles appear in this problem: $\cos \theta_{\vec{k}\vec{x}} = \hat{k} \cdot \hat{x}$, $\cos \theta_{\vec{k}'\vec{x}} = \hat{k}' \cdot \hat{x}$, and $\cos \theta_{\vec{k}'\vec{k}} = \hat{k}' \cdot \hat{k}$. You will have to find identities involving Legendre polynomials and spherical harmonics in order to evaluate the angular integrals that occur here.

   (b) Find an expression for $\sin \delta_\ell(k)$ to first order in $V$. This is the “first Born approximation”, $\delta_\ell^{BA}(k)$.

   (c) Is $|\sin \delta_\ell^{BA}(k)|$ less than unity? If not, why?

2. Scattering from a atom or a nucleus in the first Born approximation

   When low-energy electrons are scattered off atoms, the motion is non-relativistic and the atomic recoil can be neglected. If the atom is left in its ground state, where it began, the scattering is called elastic. Neglect spin. The cross section for scattering from incoming momentum $\vec{k}$ to outgoing momentum $\vec{k}'$ can equally well be regarded as function of the angles $\theta$ and $\phi$ defined relative to the $\hat{k}$ axis.

   (a) What is the relation between the momentum transfer $q \equiv |\vec{q}| = |\vec{k} - \vec{k}'|$ and the angles $\theta$ and $\phi$?
(b) Show that in the first Born approximation, the differential cross section is given by
\[
\frac{d\sigma}{d\phi d\cos \theta} = \left( \frac{2}{a_0 q^2} \right)^2 (Z - F(\vec{q}))^2
\]
where \(a_0 = \frac{\hbar^2}{m e^2}\). \(F(\vec{q})\) is known as the form factor and contains all the information, measurable in elastic scattering, about the atomic electron charge distribution, \(\rho(\vec{x})\).

\[
F(\vec{q}) = \int d^3 x e^{i\vec{q} \cdot \vec{x}} \rho(\vec{x})
\]

(c) Assuming that the charge density is spherically symmetric, \(\rho(\vec{x}) = \rho(r)\), note \(F(0) = Z\), and show that
\[
\left. \frac{dF}{dq^2} \right|_{q=0} = -\frac{1}{6} \langle r^2 \rangle,
\]
where \(\langle r^2 \rangle\) is the mean square charge radius of the atom. What is the forward \((\theta \to 0)\) limit of the differential cross section?

(d) For hydrogen \(\rho(r) = \frac{1}{\pi a_0} e^{-2r/a_0}\). Find \(F(q)\) and \(\langle r^2 \rangle\).

Now consider scattering from a nucleus in the same approximation: non-relativistic, no spin, no recoil.

(e) Nuclei have rather sharp surfaces. Model a nucleus with charge \(Z\) and mass number \(A\) as a uniform spherical charge distribution \(\rho(r, Z, A) = \rho_0(Z, A)\) for \(r \leq R(A)\) and \(\rho = 0\) for \(r > R(A)\). A good estimate for \(R(A)\) is \(R(A) = R_0 A^{1/3}\), with \(R_0 = 1.2\) fm. Compute the cross section for scattering in the first Born approximation as a function of \(Z\) and \(A\).

(f) Plot the results of the previous section as a function of \(\theta\) at fixed electron energy. Choose a few values of electron energy that show the characteristic structure of the cross section.

(g) Real nuclei do not have sharp surfaces. Can you describe qualitatively what happens to the cross section you computed in the previous part if the surface is smeared out over \(\delta R \ll R(A)\)?

3. Born Approximation for a \(\delta\)-shell

Consider scattering from a \(\delta\)-shell potential,
\[
2mV(r) = -\lambda \delta(r - a).
\]

(a) Find the first Born approximation to the scattering amplitude, \(f(k, \theta)\).

(b) Sketch \(f(k, \theta)\) as a function of \(\theta\) at fixed \(k\).

(c) Extract from \(f(k, \theta)\) the first Born approximation to the \(s\)-wave phase shift, \(\delta_0(k)\).

(d) Compute the \(s\)-wave phase shift directly by matching interior and exterior solutions. How good is the Born approximation?
4. Spin dependent scattering

A particle with spin-1/2 (mass $m$) scatters from a spin dependent potential. Suppose the spin “up” state is subject to a potential $V(\vec{x})$, while the spin “down” state is subject to no potential. In addition, suppose there is a “spin flip” potential $W(\vec{x})$ which connects spin up and down, so

$$V = \begin{pmatrix} V(\vec{x}) & W(\vec{x}) \\ W(\vec{x}) & 0 \end{pmatrix}$$

in matrix notation. Assume $V$ to be real, and assume both $V(\vec{x})$ and $W(\vec{x})$ to be short range.

(a) Write down the Schrödinger equation (in terms of a two-component spinor wavefunction) for motion in this potential.

(b) Introduce a spin index into the notation for scattering states:

$$|\phi_{\vec{k}}\rangle \rightarrow |\phi_{\vec{k},s}\rangle \quad s = \uparrow \text{ or } \downarrow$$

$$|\psi^{(+)}_{\vec{k}}\rangle \rightarrow |\psi^{(+)}_{\vec{k},s}\rangle$$

and write the Lippmann–Schwinger integral equation for $|\psi^{(+)}_{\vec{k},s}\rangle$.

(c) Calculate the scattering amplitude

$$f_{s,s'}(\vec{k},\vec{k}') = -\frac{2m}{4\pi^2}(\vec{k}' s' | V | \psi^{(+)}_{\vec{k},s})$$

to lowest non-trivial order in the Born expansion for all four values of $(s,s')$. Express your answer in terms of the Fourier transforms of $V(\vec{x})$ and $W(\vec{x})$. Note that the expansion for $f_{\uparrow\uparrow}$ begins with a term quadratic in $V$. Simplify these results in the case that $V$ and $W$ depend only on $r = |\vec{x}|$. 