Looking Back: Conformal Newtonian Gauge

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Conformal Newtonian Gauge

The metric:

\[ ds^2 = -(1 + 2\Phi)dt^2 + a^2(t) \left[ (1 - 2\Psi)\delta_{ij} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij} \right] \ dx^i \ dx^j , \]

where

\[ \frac{\partial C_i}{\partial x^i} = 0 , \quad \frac{\partial D_{ij}}{\partial x^i} = 0 , \quad D_{ii} = 0 . \]

Leads to a Poisson-like equation, Weinberg’s (5.3.26):

\[ \frac{1}{a^2} \nabla^2 \Psi = 4\pi G \delta \rho - 12\pi GH(\bar{\rho} + \bar{\rho}) \delta u . \]

**Question:** What is the significance of the 2nd term on the right?

Note that \( \delta u \) is not even local! \( \delta u_i = \frac{\partial \delta u}{\partial x^i} + \delta u^V_i \), where \( \frac{\partial \delta u^V_i}{\partial x^i} = 0. \)
Newtonian Gauge: Einstein Equations

$R_{00}$ equation:

$$\frac{1}{a^2} \nabla^2 \Phi + \frac{6\ddot{a}}{a} \Phi + \frac{3\dot{a}}{a} (\dot{\Phi} + 2\dot{\Psi}) + 3\ddot{\Psi} = 4\pi G (\delta \rho + 3\delta p + \delta \pi_{ii}) ,$$

where the correction to the perfect fluid energy-momentum tensor is written

$$\delta \pi_{ij} \equiv \partial_i \partial_j \pi^S + \partial_i \pi^V_j + \partial_j \pi^V_i + \pi^T_{ij} ,$$

with

$$\frac{\partial \pi^V_i}{\partial x^i} = 0 , \quad \frac{\partial \pi^T_{ij}}{\partial x^i} = 0 , \quad \pi^T_{ii} = 0 , \quad \text{so} \quad \delta \pi_{ii} = \nabla^2 \pi^S .$$

Note that the LHS of the $R_{00}$ equation is not $R_{00}$. Perturbations from the metric in

$$T_{\mu\nu}^{\text{perfect}} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu$$

are brought to the LHS of the equation.
\( R_{0i} \) equation:

\[
2 \left( \partial_i \dot{\Psi} + \frac{\ddot{a}}{a} \partial_i \Phi \right) + \frac{1}{2} \nabla^2 \dot{C}_i = 2 \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \delta u_i .
\]

\( R_{ij} \) equation:

\[
\begin{align*}
&\left[ 2 \left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right) \Phi + \frac{\dot{a}}{a} \left( \dot{\Phi} + 6 \dot{\Psi} \right) + \ddot{\Psi} - \frac{1}{a^2} \nabla^2 \Psi \right) \delta_{ij} + \frac{1}{a^2} \partial_i \partial_j (\Phi - \Psi) \\
&- \frac{1}{2} \left( \partial_i \ddot{C}_j + \partial_j \ddot{C}_i \right) - \frac{3}{2} \frac{\dot{a}}{a} \left( \partial_i \dot{C}_j + \partial_j \dot{C}_i \right) \\
&- \frac{1}{2} \left( \dddot{D}_{ij} + 3 \frac{\dot{a}}{a} \dot{D}_{ij} - \frac{1}{a^2} \nabla^2 D_{ij} \right) = 4\pi G (\delta p - \delta \rho + \delta \pi_{ii}) \delta_{ij} - 8\pi G \delta \pi_{ij} .
\end{align*}
\]

Trace equation \((R_{ii})\):

\[
3 \left[ 2 \left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right) \Phi + \frac{\dot{a}}{a} \left( \dot{\Phi} + 6 \dot{\Psi} \right) + \ddot{\Psi} - \frac{1}{a^2} \nabla^2 \Psi \right] + \frac{1}{a^2} \nabla^2 (\Phi - \Psi)
\]

\[
= 4\pi G (3 \delta p - 3 \delta \rho + \delta \pi_{ii}) .
\]
\[ R_{00} \text{ equation:} \]
\[
\frac{1}{a^2} \nabla^2 \Phi + \frac{6\ddot{a}}{a} \Phi + \frac{3\dot{a}}{a} (\dot{\Phi} + 2\dot{\Psi}) + 3\ddot{\Psi} = 4\pi G (\delta \rho + 3\delta \rho + \delta \pi_{ii}) .
\]

Rewrite of \( R_{ii} \) equation:
\[
6 \left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} \right) \Phi + 3 \frac{\dot{a}}{a} (\dot{\Phi} + 6\dot{\Psi}) + 3\ddot{\Psi} + \frac{1}{a^2} \nabla^2 (\Phi - 4\Psi)
\]
\[
= 4\pi G (3 \delta \rho - 3\delta \rho + \delta \pi_{ii}) .
\]

Subtracting \( R_{ii} \) equation from \( R_{00} \) equation and dividing by 4:
\[
\frac{1}{a^2} \nabla^2 \Psi - 3H (\dot{\Psi} + H \Phi) = 4\pi G \delta \rho ,
\]
where \( H = \dot{a}/a \).
The extra term is related to Weinberg’s expression through the $R_{0i}$ equation:

$$2 \left( \partial_i \dot{\Psi} + \frac{\dot{a}}{a} \partial_i \Phi \right) + \frac{1}{2} \nabla^2 \dot{C}_i = 2 \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \delta u_i \quad \Rightarrow \text{extract scalar part}$$

$$\partial_i (\dot{\Psi} + H \Phi) = \dot{H} \partial_i \delta u \quad \Rightarrow$$

$$\dot{\Psi} + H \Phi = \dot{H} \delta u = -4\pi G (\bar{\rho} + \bar{p}) \delta u ,$$

so

$$\frac{1}{a^2} \nabla^2 \Psi = 4\pi G \delta \rho - 12\pi GH (\bar{\rho} + \bar{p}) \delta u .$$
The extra term is related to Weinberg’s expression through the $R_{0i}$ equation:

$$2 \left( \partial_i \dot{\Psi} + \frac{\dot{a}}{a} \partial_i \Phi \right) + \frac{1}{2} \nabla^2 \dot{C}_i = 2 \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \delta u_i \quad \Longleftrightarrow \quad \text{extract scalar part}$$

$$\partial_i (\dot{\Psi} + H \Phi) = \dot{H} \partial_i \delta u \quad \Longrightarrow \quad (\dot{\Psi} + H \Phi) = \dot{H} \delta u = -4\pi G (\bar{\rho} + \bar{p}) \delta u ,$$

so

$$\frac{1}{a^2} \nabla^2 \Psi = 4\pi G \delta \rho - 12\pi GH (\bar{\rho} + \bar{p}) \delta u .$$

But why is this extra term here!?
Look at the divergence of the fluid velocity:

\[ u^\mu;\mu = \frac{\partial u^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu \lambda} u^\lambda \]

\[ = 3 \bar{H} + \frac{1}{a^2} \partial_i u_i + \frac{3}{2} \bar{H}h_{00} - \frac{1}{a^2} \bar{H}h_{ii} + \frac{1}{2a^2} \dot{h}_{ii} - \frac{1}{2a^2} \partial_i h_{i0} . \]

In Newtonian gauge,

\[ u^\mu;\mu = 3 \bar{H} + \frac{1}{a^2} \partial_i u_i - 3(\ddot{\Phi} + H\dot{\Phi}) . \]
You may recall that on 4/20/09 we used this relation to show that the perturbation variable

\[ \mathcal{R}_q \equiv -\Psi_q + H \delta u_q \]

can be related to the variable

\[ K \equiv a^2 \left[ \frac{8\pi G \rho_{1oc} - H_{1oc}^2}{3} \right] , \]

where \( \rho_{1oc} \) is the local energy density \( \bar{\rho} + \delta \rho \), and \( H_{1oc} = \frac{1}{3} u^\mu ; \mu \) is the local expansion rate for the comoving fluid. Note that \( K \) is a local version of the Robertson-Walker curvature constant \( k \), so it is immediately apparent that \( K \) is gauge-invariant, since any scalar which is constant in the background solution is gauge-invariant. It is also apparent that it is conserved in the long wavelength limit, since it is constant in a homogeneous solution. \( K \) was found to be related to \( \mathcal{R} \) by

\[ K = -\frac{2}{3} \nabla^2 \mathcal{R} . \]
For points at rest in the coordinate system,

\[ u^\mu;\mu = 3\ddot{H} - 3(\dot{\Psi} + H\Phi) = 3(\ddot{H} + \delta H_{\text{coord}}), \]

where \( \delta H_{\text{coord}} = -(\dot{\Psi} + H\Phi) \) is the perturbation in the local Hubble expansion rate for points at rest in the coordinate system. Since the local Hubble rate is perturbed, so is the local critical density:

\[ \rho_{\text{cr}} = \frac{3H^2}{8\pi G} \quad \implies \quad \delta \rho_{\text{cr}} = \frac{3H \delta H}{4\pi G} = -\frac{3H}{4\pi G}(\dot{\Psi} + H\Phi). \]

So the Poisson equation can be written

\[ \frac{1}{a^2} \nabla^2 \Psi = 4\pi G \delta \rho + 3H(\dot{\Psi} + H\Phi) \]

\[ = 4\pi G(\delta \rho - \delta \rho_{\text{cr}}). \]

Thus the equation is telling us that the Newtonian potential responds to perturbations in the mass density relative to the local critical density.