Adaptive Control of Nonlinearly Parameterized Systems with a Triangular Structure *

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Abstract

This paper deals with adaptive control of a class of nonlinear systems with a triangular structure and nonlinear parameterization. In [9] it was shown that a class of second-order nonlinearly parameterized systems can be adaptively controlled in a globally stable manner. In this paper, we extend our approach to all nth order systems that have a triangular structure. Global boundedness and convergence to within a desired precision ε is established for both regulation and tracking. Extensions to cascaded systems containing linear dynamics and static nonlinearities are also presented.

1 Introduction

One of the most common assumptions made in the context of adaptive control is that the unknown parameters occur linearly, and appear in linear [13] and nonlinear systems [6, 11, 16, 17]. Recently, a new approach has been developed [1, 12, 9, 2] to address nonlinearly parameterized (NLP) systems and their adaptive control. The main problem that is introduced due to nonlinearity in the parameterization is the failure of the gradient approach. Whether viewed from an optimization or a stabilization view-point, the gradient scheme is a powerful and simple procedure for adapting the adjustable parameters to cope with parametric uncertainty. When parameters occur linearly, the gradient scheme is sufficient to minimize the underlying cost function related to the parameter error; the gradient scheme guarantees a quadratic Lyapunov function leading to global stability. These properties are not sufficient when parameters occur nonlinearly. The approach in [1, 12, 9, 2] outlines the construction of an alternative strategy for generating adaptation laws that guarantee stability. In [1], it is assumed that the underlying parameterization is convex/concave which is

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made use of in constructing a quadratic Lyapunov function. In [12], the results are extended to include general parameterizations. In both cases, it is assumed that state variables are accessible and that the underlying class of nonlinear systems are of the form

\[ \dot{X}_p = A_p X_p + b(f(\phi(t), \theta) + u) \]  

(1)

where \( f \) is a scalar nonlinearity in the unknown parameter \( \theta \), can be globally stabilized. In [9], the class in (1) is extended further to include a special class of systems where matching conditions [18] are not satisfied. These systems are second-order, have a triangular structure, and are of the form

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) + \sum_{i=1}^{n} \sigma_i f_i(x_1(t), \theta_i) \\
\dot{x}_2(t) &= f_0(x, \theta_j) + u(t)
\end{align*}
\]  

(2)

where \( x = [x_1, x_2]^T \), \( x_1, x_2 \), and \( u \) are scalar functions of time, \( \sigma_i, \theta_i \), and \( \theta_j \) are scalar parameters which are unknown, and \( f_0: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, f_i: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, ..., n \). Global stabilization and tracking to within a desired precision is established in [9].

In this paper, we seek to generalize these results to systems with a triangular structure and are of arbitrary order. These systems can be described as

\[
\begin{align*}
\dot{x}_1 &= \gamma_1(x_2) + f_1(x_1, \theta_1) \\
\dot{x}_2 &= \gamma_2(x_3) + f_2(x_1, x_2, \theta_2) \\
&\vdots
\end{align*}
\]  

(3)

\[
\begin{align*}
\dot{x}_n &= u + f_n(x_1, x_2, \ldots, x_n, \theta_n)
\end{align*}
\]

where \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n, \theta_i \in \mathbb{R}^{m_i} \), and \( \theta_i \) are unknown parameters. The goal is for \( x_1 \) to track a desired trajectory \( x_d \). It is assumed that the states are accessible, and that the unknown parameters belong to a compact set \( \Theta_m \) in \( \mathbb{R}^{m_c} \). It is also assumed that \( \gamma_i \) and \( f_i \) are differentiable functions of their arguments, and that \( \frac{\partial \gamma_i}{\partial x_i} \) does not vanish.

The proposed adaptive controller provides a stability framework for estimating the unknown parameters. For this purpose, in addition to a stability framework, functions generated using a min-max optimization problem are introduced in the adaptive controller, as in [1, 12].

The paper is organized as follows. Section 2 introduces a few necessary preliminaries about NLP systems. In section 3, we present the adaptive tracking controller for the systems of the form in (3). The closed-loop global stability of the proposed controller is established. A direct extension of the proposed controller to a class of LNL systems is presented in section 4. Final remarks are offered in section 5.
2 Preliminaries

The following definitions and lemmas are useful in the development of the adaptive controllers. For definitions of concave/convex functions, see [1]. The notation \( x_{[i,j]} \) for \( j \geq i \) is used as shorthand to represent the elements \( x_i, x_{i+1}, \ldots, x_j \) of the vector \( x \in \mathbb{R}^n, n \geq j-i+1 \). Also, the functions \( \text{sat}(x) : \mathbb{R}^n \to \mathbb{R}^n, \sigma(x) : \mathbb{R}^n \to \mathbb{R}^n \) are defined as

\[
\text{sat}(x) = \begin{cases} 
\frac{x}{\|x\|}, & \|x\| > 1 \\
x, & \|x\| \leq 1
\end{cases} ; \quad \sigma(x) = \begin{cases} 
\frac{x}{\|x\|}, & \|x\| > 0 \\
0, & \|x\| = 0
\end{cases}
\]

(4)

**Definition 1** A smoothing function \( S(z, \epsilon) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is an \((n-1)\) times differentiable odd function which satisfies the following:

\[
\sigma(S(z, \epsilon)) = \sigma(z); \quad S(z, \epsilon) = \sigma(z) \quad \forall |z| \geq \epsilon > 0; \quad |S(z, \epsilon)| \leq 1. \tag{5}
\]

One example of \( S(z, \epsilon) \) is a \( \text{sat}(z) \) function with smooth corners.

**Definition 2** \( \overline{f}(x, z, \epsilon) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is said to be a smooth bounding function of \( f(x, \theta) \) with respect to \( z \) and a buffer \( \delta_f \) if

\[
\overline{f}(x, z, \epsilon) = S(z, \epsilon) \left( \max_{\theta \in \Theta} |f(x, \theta)| + \delta_f \right).
\]

(6)

**Lemma 1** If \( \overline{f}(x, z, \epsilon_0) \) is a bounding function of \( f(x, \theta) \) with respect to \( z \) with a buffer \( \delta_f > \delta \), then for any \( y, \) all \( \theta \in \Theta \), a compact set in \( \mathbb{R}^m \), and for all \( |z| > \epsilon_0 \), the following holds:

\[
\sigma(z) \left[ (f(x, \theta) - \overline{f}(x, z, \epsilon_0)) + \delta S(y, \epsilon_0) \right] \leq -(\delta_f - \delta) < 0. \tag{7}
\]

**Proof:** The proof follows in a straightforward manner by substituting (6) into (7) and noting that eq.(5) implies that \( \sigma(z)S(y, \epsilon_0) \leq 1 \) and since \( |z| > \epsilon_0 \), \( S(z, \epsilon_0) \sigma(z) = 1 \). \( \bullet \)

**Lemma 2** Let \( \Theta \) be a simplex in \( \mathbb{R}^m \) whose \( m+1 \) vertices are \( \theta_{S_i}, i = 1, \ldots, m+1 \). Let \( \theta \in \Theta \), and for a given \( \hat{\theta} \in \Theta \), let

\[
J(\omega, \theta) = \beta \left[ f(\phi, \theta) - f(\phi, \hat{\theta}) + \omega(\hat{\theta} - \theta) \right] ;
\]

(8)

\[
a_0 = \min_{\omega \in \mathbb{R}} \max_{\theta \in \Theta} J(\omega, \theta); \quad \omega_0 = \arg \min_{\omega \in \mathbb{R}} \max_{\theta \in \Theta} J(\omega, \theta), \tag{9}
\]

where \( \beta \) and \( \phi \) are known quantities independent of \( \theta \). Then, given \( \phi \) and \( \beta \), and defining \( g(\theta) = \text{sign}(\beta)f(\phi, \theta) \),

\[
a_0 = \begin{cases} 
A_1, & \text{if } g(\theta) \text{ is convex on } \Theta \\
0, & \text{if } g(\theta) \text{ is concave on } \Theta
\end{cases} \quad \omega_0 = \begin{cases} 
A_2, & \text{if } g(\theta) \text{ is convex on } \Theta \\
\nabla f_{\hat{\theta}}, & \text{if } g(\theta) \text{ is concave on } \Theta
\end{cases}
\]

(10)
where \( A = [A_1, A_2]^T = G^{-1} b \), \( A_1 \) is a scalar, \( A_2 \in \mathbb{R}^m \),

\[
G = \begin{bmatrix}
-1 & \beta(\hat{\theta} - \theta_{s_1})^T \\
-1 & \beta(\hat{\theta} - \theta_{s_2})^T \\
\vdots & \vdots \\
-1 & \beta(\hat{\theta} - \theta_{s_{m+1}})^T
\end{bmatrix}, \quad b = \begin{bmatrix}
\beta(\hat{f} - f_{s_1}) \\
\beta(\hat{f} - f_{s_2}) \\
\vdots \\
\beta(\hat{f} - f_{s_{m+1}})
\end{bmatrix}.
\] (11)

and \( \hat{f} = f(\phi, \hat{\theta}), f_{S_i} = f(\phi, \theta_{S_i}) \).

Proof: See [1, 12].

**Lemma 3** Let \( \alpha, \epsilon \) be arbitrary positive quantities, and let \( \alpha_{\text{max}} \geq \alpha \). For a given \( \hat{\theta} \in \Theta \subset \mathbb{R}^m \), let \( a_0 \) and \( \omega_0 \) be chosen as in eq (9) with \( \beta = \alpha_{\text{max}} \text{sign}(z) \), \( z \in \mathbb{R} \). If \( |z| \geq \epsilon \), the following is then true \( \forall \phi \) and \( \forall \theta \in \Theta \), whether \( f(\phi, \theta) \) is concave or convex on \( \Theta \):

\[
z \left\{ \alpha \left[ f(\phi, \theta) - f(\phi, \hat{\theta}) + (\hat{\theta} - \theta)^T \omega_0 \right] - a_0 \text{sat} \left( \frac{z}{\epsilon} \right) \right\} \leq 0
\] (12)

where the function \( \text{sat}(\cdot) \) denotes the saturation function.

Proof: See [9].

### 3 The controller structure

This section outlines the basic ideas behind our approach to designing a controller for the system in (3). In [9], we have derived a stabilizing controller for (2) when \( n = 2 \). The question therefore pertains to the complexities introduced by a higher order system. This section discusses these complexities, and how they can be addressed, with the starting point being the approach taken in [9].

Briefly, the problem considered in [9] is the stabilization of the system

\[
\begin{align*}
\dot{z}_1 &= z_2 + f_1(z_1, \theta) \\
\dot{z}_2 &= u
\end{align*}
\] (13)

when \( \theta \) is unknown. One of two main obstacles here is that the dynamics of \( z_1 \) are not under our direct control. The other obstacle is that the unknown parameter \( \theta \) appears nonlinearly in the dynamics of \( z_1 \). We overcome the first obstacle by choosing errors \( e_0 \) and \( e_1 \) such that when \( e_1 \to 0 \) it assures that \( e_0 \to 0 \). The task that remains then is to choose a \( u \) such that \( e_1 \) tends to zero. In particular, a choice of

\[
e_0 = z_1, \quad e_1 = z_2 + g_1(z_1, e_0)
\] (14)
leads to error equations

\[ \dot{e}_0 = e_1 + f_1(z_1, \theta) - g_1(z_1, e_0) \]
\[ \dot{e}_1 = u + f_2(z_1, z_2, \theta) \]

where \( f_2 = \frac{\partial g_1}{\partial z_1}(z_2 + f_1) + \frac{\partial g}{\partial e_0}(e_1 + f_1 - g_1) \). By making \( g_1 \) a bounding function of \( f_1 \) with respect to \( e_0 \), we essentially stabilize \( e_0 \) in the absence of \( e_1 \). To overcome the second obstacle and choose \( u \) such that \( e_1 \to 0 \), especially due to uncertainties in a nonlinear parameterization, the min-max algorithm as in [1] is used.

The problem that we consider in this paper is for the state \( x_1 \) in (3) to asymptotically track a desired trajectory \( x_d \). We assume that \( x_d \) is sufficiently smooth, so that \( \dot{x}_d, x_d^{[2]}, \ldots, x_d^{[n]} \) are bounded. A direct extension of our stabilization approach in [9] to a higher-dimensional tracking problem for the system in (3) requires appropriate characterizations of \( n \) errors, \( e_j, j = 0, 1, \ldots, n - 1 \). These errors are chosen such that, due to the characteristics of the system in (3), the control input \( u \) directly appears only in the dynamics of \( e_{n-1} \). The basic idea is to define the other errors, \( e_j, j = 0, 1, \ldots, n - 2 \) in such a way that if \( e_{n-1} \to 0 \), it guarantees that all errors \( e_j, j = 0, \ldots, n - 1 \), tend to zero. Let us assume that the errors are of the form

\[ e_0 = x_1, \quad e_j = \psi_j(x_{[1,j+1]}, x_d^{[j]}), \quad j = 1, \ldots, n - 1. \]

Suppose that the functions \( \psi_j \) are chosen in such a way that these errors satisfy the relationship

\[ \dot{e}_j = e_{j+1} - e_j + f_{j+1}(x_{[1,j+1]}, \theta) - \overline{f}_{j+1}(x_{[1,j]}, e_{[0,j-1]}), \quad j = 0, \ldots, n - 2 \]
\[ \dot{e}_{n-1} = u + f_n(x_{[1,n]}, \theta) \]

where \( e_{-1} = 0 \), for suitably defined \( f_j \) and \( \overline{f}_j \). The advantage of the structure in (17)-(18) is apparent if \( \overline{f}_j \) is a bounding function of \( f_i \) with respect to \( e_{i-1} \), since the latter leads to the property

\[ (f_{i+1} - \overline{f}_{i+1})\sigma(e_i) \leq 0. \]

This follows since \( V = \frac{1}{2} \sum_{i=0}^{n-1} e_i^2 \) yields a time derivative of the form

\[ \dot{V} = \sum_{i=0}^{n-2} \left[ e_i(e_{i+1} - e_{i-1}) + e_i \left( f_{i+1} - \overline{f}_{i+1} \right) \right] + e_{n-1}(u + f_n) \leq e_{n-1}(e_{n-2} + u + f_n) \]

which suggests that \( V \) is a Control Lyapunov Function for (17)-(18), leading to global stabilization.
The question that arises is if indeed errors $e_j$ can be constructed as in (16) so that (a) they satisfy (17)-(18) and (b) ensure that the resulting controller has no discontinuities. This is answered by the following recursive relationships:

\[
\begin{align*}
e_0 &= x_1 - x_d \\
e_i &= e_{i-2} + \Gamma_i (x_{[2,i]} \gamma_i (x_{i+1}) + g_i (x_{[1,i]} - x_d^{(i)}), & i = 1, \ldots, n - 1 \quad (21)
\end{align*}
\]

where

\[
\begin{align*}
g_i (x_{[1,i]}, e_{[0,i-1]}) &= k_{i-1} + \overline{T}_{i-1} \quad (22) \\
k_i (x_{[1,i+1]}, e_{[0,i]}) &= e_{i-1} - e_{i-3} - \overline{T}_{i-2} + \sum_{j=2}^{i} \frac{\partial \Gamma_{i-1}}{\partial x_j} \gamma_j \gamma_i + \sum_{j=1}^{i} \left( \frac{\partial k_{i-1}}{\partial x_j} + \frac{\partial \overline{T}_{i-1}}{\partial x_j} \right) \gamma_j + \sum_{j=0}^{i-1} \left( \frac{\partial k_{i-1}}{\partial e_j} + \frac{\partial \overline{T}_{i-1}}{\partial e_j} \right) (e_{j+1} - e_{j-1} - \overline{T}_{j}) \quad (23) \\
h_i (x_{[1,i+1]}, e_{[0,i]}, \theta_{[1,i+1]}) &= h_{i-2} + \sum_{j=2}^{i} \frac{\partial \Gamma_{i-1}}{\partial x_j} f_j (x_{[1,j]}, \theta_k) \gamma_i + \Gamma_i f_{i+1} (x_{[1,i+1]}, \theta_{i+1}) + \sum_{j=0}^{i-1} \left( \frac{\partial k_{i-1}}{\partial e_j} + \frac{\partial \overline{T}_{i-1}}{\partial e_j} \right) h_j \quad (24)
\end{align*}
\]

for $i = 2, \ldots, n - 1$,

\[
\Gamma_i = \Gamma_{i-1} \frac{\partial \gamma_i}{\partial x_{i+1}}, \quad i = 1, \ldots, n - 1 \quad (25)
\]

with

\[
k_0 = 0; \quad h_{-1} = 0; \quad h_0 = f_1 (x_1, \theta); \quad \Gamma_0 = 1, \quad e_{-1} = 0.
\]

In (22)-(24), $\overline{T}_i (x_{[1,i+1]})$ are chosen as smooth bounding functions of $h_i$ with respect to $e_i$, with buffers $\delta_i$ such that $\delta_f \geq e_i$ and $\delta_{i-1} - e_{i-1} + e_{i+1}$ for $i = 2, \ldots, n - 2$. Algebraic manipulations of (21)-(24) can be carried out to show that $e_i$’s satisfy (17)-(18). We now propose an adaptive controller for the tracking problem when $\theta$ is unknown. For this purpose we note that the recursive relation in (24) implies that the function $h_i$ can be cast in the form

\[
\begin{align*}
h_i &= \sum_{j=1}^{i+1} h_{ij} (x_{[1,j]}, e_{[0,j-1]}, \theta_{[1,i+1]}) = \sum_{j=1}^{i+1} \phi_{ij} (x_{[1,j]}, e_{[0,j-1]}) f_j (x_{[1,j]}, \theta_j), \quad (26)
\end{align*}
\]

where $\phi_{ij}$ is a known function of its arguments. Since the arguments of $\phi_{ij}$ are themselves functions of time only, from (26) it follows that the dependence of $h_i$ on the unknown parameter $\theta_i$ is equivalent to the dependence of $f_i$ on $\theta_i$. Therefore, any initial assumptions
on the convexity/concavity of $f_i$ with respect to the unknown parameter $\theta_i$ are preserved in $h_i$. The adaptive controller is given by

\begin{align}
\hat{u} & = \Gamma_{n-1}^{-1}(c^T \epsilon_{n-1} - \hat{f}_n - a^* S(\epsilon_{n-1}, \epsilon_{n-1}) + x^n_d) \quad c > 0 \quad (27) \\
\hat{f}_n & = s_{n-1}(x_{[1,n]}, \hat{\theta}_{[1,n]}) \\
\hat{h}_{n-1} & = \sum_{j=1}^{n} \phi_{n-1,j}(x_{[1,j]}, \epsilon_{[0,j-1]}) f_j(x_{[1,j]}, \hat{\theta}_j) \\
\hat{\theta}_j & = \epsilon_i^T \omega_i^*_j \\
\epsilon_i^* & = \epsilon_i - \epsilon_i \text{sat}(\epsilon_i/\epsilon_i), \quad \epsilon_i > 0, \quad i = 0, \ldots, n-1 \quad (29) \\
a^* & = \sum_{j=1}^{n} \omega_j^* \\
(a_j^*, \omega_j^*) & = \min_{\omega_j} \max_{\theta_j} \left[ h_{n-1,j-1}(x, \theta_j) - h_{n-1,j-1}(x, \hat{\theta}_j) + (\hat{\theta}_j - \theta_j) \omega_j \right]. \quad (30)
\end{align}

As mentioned earlier in Lemma 2, the closed-form solutions to (30) can be found along the lines outlined in [1, 12], with the solutions being considerably simpler when $h_{n-1,j}$ is convex/concave with respect to $\theta_j$.

The stabilizing property of the controller in (27)-(30) is given in the following theorem.

**Theorem 1** For the system in (3), the controller defined by (21)-(24), (27)-(30) results in each $|e_i(t)|$ tending to $\epsilon_i$ as $t \to \infty$ thereby achieving global stability and desired tracking performance.

**Proof:** By differentiating with time (21), and with the functions $k_i$ as in (23) and $h_i$ as in (24), it can be shown that

\begin{equation}
\dot{e}_i = e_{i+1} - e_{i-1} + h_i - \hat{h}_i, \quad i = 0, \ldots, n-2. \quad (31)
\end{equation}

Choosing $V = \frac{1}{2} \sum_{i=0}^{n-1} \epsilon_i^T \dot{\epsilon}_i^T + \overline{\epsilon}^T \overline{\dot{\epsilon}}$ and using Eqs. (21), (29), and (31), and the fact that

\begin{equation}
\frac{d}{dt} \left( \epsilon_i^T \right) = 2 \epsilon_i^T \dot{\epsilon}_i, \quad \text{we obtain a time derivative}
\end{equation}

\begin{equation}
V = \sum_{i=0}^{n-2} \epsilon_i^T (e_{i+1} - e_{i-1} + h_i - \hat{h}_i) + \epsilon_{n-1}^T (\Gamma_{n-1} u + f_n - x^n_d) - \epsilon_{n-1}^T \overline{\epsilon}^T \dot{\epsilon}^* \quad (32)
\end{equation}

where $f_n = k_{n-1} + h_{n-1}$. Substituting the control law from (27) into (32), we have

\begin{align}
V & = \sum_{i=0}^{n-2} \epsilon_i^T (e_{i+1} - e_{i-1} + e_{i+1} \text{sat}(y_{i+1}) - e_{i-1} \text{sat}(y_{i-1}) + h_i - \hat{h}_i) \\
& - \epsilon_{n-2}^T \epsilon_{n-1}^* - c \epsilon_{n-1}^T \epsilon_{n-1} + \epsilon_{n-1}^T \left( f_n - \hat{f}_n - \overline{\epsilon}^T \dot{\epsilon}^* - a^* S(\epsilon_{n-1}, \epsilon) \right)
\end{align}
where \( y_i = e_i / \epsilon_i \). Therefore,

\[
\dot{V} \leq -c \epsilon'_n e_{n-1}^2 + \sum_{i=0}^{n-2} e'_i \left( h_i - \overline{h}_i + \epsilon_{i+1}sat(y_{i+1}) - \epsilon_{i-1}sat(y_{i-1}) \right).
\] (33)

Since \( \delta_{f_0} \geq \epsilon_1 \) and \( \delta_{\hat{f}_i} \geq \epsilon_{i-1} + \epsilon_{i+1} \) and \( \overline{h}_i \) are bounding functions of \( h_i \), with buffers \( \delta_{\hat{f}_i} \), respectively, it follows that the second term is nonpositive. Hence,

\[
\dot{V} \leq 0.
\]

Since \( e'_i \) are bounded, and \( \Gamma_i, \gamma_i, g_i \) are bounded functions of their arguments, it follows that \( x_i \) are bounded. By Barbalat’s lemma this implies that all \( e'_i \) tend to zero. This means that all \( |e_i| \) tend to \( \epsilon_i \), which in turn, sets the bounds on all \( x_i \). 

Comments:

1. We note that the main result, stated above, establishes global boundedness and tracking to within a desired precision, in contrast to local results obtained in the literature (for example, [5]).

2. The stabilizing controller proposed here requires the availability of two functions \( a^* \) and \( \omega^* \). These in turn imply that closed-form solutions of (30) are needed. These can be constructed in a simple manner, as outlined in [1], when \( f_n \) is a convex/concave function of \( \theta \). Convexity/concavity of the underlying nonlinearity has also been exploited in [4, 15]. The computational burden increases when \( f_n \) is a general function, and is discussed in [12]. Special classes of functions \( f_i \) which can be reparameterized so as to result in concavity (or convexity) are described in [14]. For all such functions, the controller proposed above results in global boundedness.

3. The proof of theorem 1 and the preceding discussions also demonstrate that stabilization of systems in chain form can be accomplished without adapting to the parameter \( \theta \). Instead of estimating \( f_n \) as \( \hat{f}_n \), one could simply construct yet another bounding function and stabilize (3). However, the advantage of using \( \hat{f}_n \) is that it enables the unknown parameter to be estimated in addition to stabilization. Once such a stable framework is generated for parameter estimation, conditions related to persistent excitation can be invoked to obtain parameter convergence. In [8], it has been shown that for a class of error models of the form

\[
\dot{e} = -e + f(\phi, \theta) - f(\phi, \hat{\theta}) - a^* S(e, \epsilon) 
\] (34)

\[
\dot{\hat{\theta}} = e' \omega^* 
\] (35)

conditions of persistent excitation of \( \phi \) with respect to \( f \) for nonlinearly parameterized systems can be derived so as to result in the convergence of the parameter \( \hat{\theta} \) to \( \theta \) to
within a desired precision $\varepsilon$. It is worth noting that the error $e_{n-1}$ satisfies a differential equation that is quite similar in form to that of (34). Hence, an extension of the result in [8] to parameter convergence using the controller presented here is quite feasible.

4. The stability result in Theorem 1 can be viewed as an extension of the parametric-strict-feedback systems considered in [6] to the case when the unknown parameters occur nonlinearly. Examples of such systems abound in several applications [1, 2, 14, 3]. In contrast to the back-stepping approach suggested in [6], we use a Bounding Function to generate the errors $e_i$ in the system. We note that in contrast to linear adaptive control where parameter adaptation is proposed at each of the $i$th level of back-stepping [6, 17], we determine the adaptive laws at the final step so as to generate a Lyapunov function. This enables us to achieve a stabilizing adaptive controller without using over-parameterization of the controller.

5. Robustness to additive external disturbances in (3) can be established in a straightforward manner by modifying the adaptive law in (28) as

$$\hat{\theta}_j = e_{n-1}^j \omega^j_i - \sigma_e \hat{\theta}_j, \quad \sigma_e > 0, \quad j = 1, \ldots, n - 1$$

We refer the reader to [7] for further details.

6. The same approach of using a Bounding Function together with the min-max algorithms can be used to stabilize coupled second-order systems in chain form given by

$$\ddot{x}_1 = x_2 + f_1(x_1, \theta)$$

$$\ddot{x}_i = x_{i+1}$$

where $\ddot{x}_{n+1} = u$, and $x_i \in \mathbb{R}^p$, $i = 1, \ldots, n$. Towards this end, vector Bounding Functions along the lines of Definition 2 have to be constructed [7]. A specific example of a system in chain form is often used in nonlinear system identification and is discussed in detail in the next section.

4 Control of L-N-L systems

A special class of chain-form systems has three systems in cascade, which include Linear dynamics, followed by static Nonlinearities, and Linear dynamics, and referred to as LNL systems [10]. One such form is given by

$$x^{(m)} = f(z, \theta)$$

$$z^{(n)} = u$$

(36)
where the unknown parameter $\theta$ lies in a compact set in $\mathbb{R}^p$ and the goal is to stabilize this system and enable $x$ to track a desired trajectory. Eq. (36) can be considered to be a particular case of the general form of (3). In what follows, we present a stabilizing controller for the case when $m$ is arbitrary and $n = 1$. The approach presented can be extended in a straightforward manner to include the systems where $n \geq 2$. Define
\[
\begin{align*}
  e_0 &= D(s) \int_0^t x(\tau) d\tau \\
  e_1 &= D_1(s)[x] + g(z, e_0)
\end{align*}
\]
where $D(s) = s^m + a_1 s^{m-1} + \ldots + a_m$ is a Hurwitz polynomial, $D_1(s) = D(s) - s^m$ and $g(z, e_0)$ is a bounding function with a buffer $\delta_f + e_0$ of $f(z, \theta)$ with respect to $e_0$. Then,
\[
\begin{align*}
  \dot{e}_0 &= e_1 + f(z, \theta) - g(z, e_0) \\
  \dot{e}_1 &= \frac{\partial g}{\partial z} u + (a_1 + \frac{\partial g}{\partial e_0}) f(z, \theta) + D_2(s)[x] + \frac{\partial g}{\partial e_0} D_1(s)[x]
\end{align*}
\]
with $D_2(s) = s(D_1(s) - a_1 s^{m-1})$. The adaptive controller is designed as
\[
\begin{align*}
  u &= \left(\frac{\partial g}{\partial z}\right)^{-1} \left[ -e'_0 - c e'_1 - D_2(s)[x] - \frac{\partial g}{\partial e_0} D_1(s)[x] - (a_1 + \frac{\partial g}{\partial e_0}) f(z, \tilde{\theta}) - a^* S(e_1, \epsilon) \right], \quad c > 0 \\
  \dot{\tilde{\theta}} &= e'_1 \omega^*
\end{align*}
\]
with $a^*$ and $\omega^*$ adjusted according to the min-max algorithm. This leads to a time derivative of $V = \frac{1}{2}(e'_0^2 + e'_1^2)$ being nonpositive,
\[
\dot{V} \leq -|e'_0|\delta_f - c |e'_1|^2,
\]
indicating global stability of the system. From the definition of $e_0$, if $|e_0| \to 0$ implies that the state $x$ and all its time derivatives $x^{(i)}$, $i = 1, \ldots, m$ are bounded as well. It should be noted that in order to be able to compute the control input $u$ as in eq. (39), it is required that the inverse of $\frac{\partial g}{\partial z}$ always exist. However, $g$ is a constructed feature of the controller, and not of the physical system. Furthermore, $g$ is a bounding function constructed from (6) as
\[
\begin{align*}
  g(x, z, \epsilon) &= S(z, \epsilon) \left( \max_{\theta \in \Theta} |f(x, \theta)| + \delta_f \right).
\end{align*}
\]
$S$ is smooth in $z$, and $\delta_f$ is arbitrary such that $|\delta_f| > \delta > 0$. Hence, $g$ can be designed in such a way that $\left(\frac{\partial g}{\partial z}\right)^{-1}$ always exists.

5 Summary

In this paper, a class of nonlinear systems of the form of (3) was considered where $f_i$ are nonlinear with respect to the dynamics as well as the parameters. We show that a globally
stabilizing and tracking controller can be determined. The stability approach consists of using Bounding Functions to define errors $e_i$ in a recursive manner which are chosen such that when $e_i \to 0$, it guarantees that $e_j, j = 0, \ldots, i$ tend to zero. Finally, the control input and the adaptive law for estimating the unknown parameters are chosen so that they guarantee the convergence of $e_{n-1}$ to zero. For this purpose, a min-max strategy, originally proposed in [1] is used.

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References


