So the upper support of the human capital distribution must be lower than some multiple of its mean which is affected by $\lambda$. Note that for capitalists to always be richer than workers, the actual amount of capital each and every capitalist receives is irrelevant! This arises because lower levels of $k$ induce higher returns to capital $r$, and higher per capitalist levels of $k$ induce lower $r$. While the percentage of capitalists in the economy, $\lambda$, does affect the necessary assumption on the distribution of human capital, it simply is an artifact of the Cobb-Douglas production function that $k$ does not.

A) The ideal tax rate for capitalists satisfies (boundary conditions matter a LOT here)

$$\max_{\tau} \mathcal{R}(1-\tau) + \tau[k \lambda + (1-\lambda)\omega \bar{h}] - \nu(\tau)$$

$$\frac{\partial I}{\partial \tau} = -\mathcal{R} + \lambda \lambda k + (1-\lambda)\omega \bar{h} - \nu'(\tau) = 0$$

$$(1-\lambda)[\omega \bar{h} - \mathcal{R}] = \nu'((\tau^*_c))$$

if $\omega \bar{h} - \mathcal{R} \leq 0 \rightarrow \tau^*_c = 0$

if $\omega \bar{h} - \mathcal{R} > 0 \rightarrow \tau^*_c > 0$

The ideal tax rate for worker with human capital $h_i$ satisfies

$$\max_{\tau} \omega h_i(1-\tau) + \tau[k \lambda + (1-\lambda)\omega \bar{h}] - \nu(\tau)$$

$$\frac{\partial I}{\partial \tau} = -\omega h_i + \lambda \lambda k + (1-\lambda)\omega \bar{h} - \nu'(\tau) = 0$$

$$\omega[\bar{h} - h_i] + \lambda [\mathcal{R} - \omega \bar{h}] = \nu'(\tau^*_i)$$

if $\omega h_i \geq \lambda \mathcal{R} + (1-\lambda)\omega \bar{h} \rightarrow \tau^*_i = 0$

if $\omega h_i < \lambda \mathcal{R} + (1-\lambda)\omega \bar{h} \rightarrow \tau^*_i > 0$

Voters will have single peaked preferences as long as $\bar{h}$ is defined. As we know that $\nu(\tau)$ is convex, the first order conditions will be concave and well behaved. However not all distributions have an average. If $\bar{h}$ is undefined, then neither type will have single peaked preferences. The assumptions on $\nu(\tau)$ are hence ‘useful’ because they let us know $-\nu(\tau)$ is concave.

Now using our assumption that capitalists are always richer than workers, we know that capitalists want the lowest tax rate, followed by the smartest workers. The workers with the lowest levels of human capital want the highest tax rate. In a voting equilibrium, the preferences of the median voter will result as the equilibrium tax rate. Any individual with income $> \lambda \mathcal{R} + (1-\lambda)\omega \bar{h}$ will vote for $\tau = 0$, whereas those with income below the societal average income will vote for positive taxes. Define

$$F(h_m) = \frac{1}{2}$$
\( F(h_{\frac{1}{2}, \lambda}) = \frac{1 - \lambda}{1 - \lambda} \)

\[ F(h_{\frac{1}{2}}) = \frac{1}{2(1 - \lambda)} \]

Hence \( h_m \) represents the worker with the median level of human capital of the workers. \( h_{\frac{1}{2}, \lambda} \) denotes the worker who’s income puts them at the \((\frac{1}{2} - \lambda)\) percentile of the income distribution. \( h_{\frac{1}{2}} \) gives the human capital of the worker with the median level of income in the society.

The equilibrium tax rate will hence be
\[
\tau = 0 \text{ if } \omega h_{\frac{1}{2}} \geq \lambda r k + (1 - \lambda) \omega \bar{h} \text{ or }
\]
\[
(\omega \bar{h} - h_{\frac{1}{2}}) + \lambda [r k - \omega \bar{h}] = \nu'(r_{\frac{1}{2}}) \text{ if } \omega h_{\frac{1}{2}} < \lambda r k + (1 - \lambda) \omega \bar{h}
\]

The case where the median voter has an income less than the average income in society is the interesting one. All capitalists would prefer a lower tax rate \((\tau = 0)\), as do workers with \( h_i > \bar{h}_{\frac{1}{2}} \). However workers with \( h_i < \bar{h}_{\frac{1}{2}} \) all would prefer a higher tax rate. Since the worker with human capital \( h_{\frac{1}{2}} \) represents the median income of society, and hence the median preference over taxes, his preferred outcome is the equilibrium tax rate.

Dynamics to the system will work in two dimensions. Changes in \( k \) and \( \lambda \) will not only change the resources in society that can be taxed, but may also change the voting block who will vote for taxes.

Changes in \( k \) are the easiest to deal with. As we saw, a change in \( k \) will not affect our assumption that capitalists are richer than the richest worker. Hence a change in \( k \) only affects the equilibrium through the effect it has on the median voter’s preferences. We see from the equation above that an increase in \( k \) will have two effects on the preferred tax rate of the median voter. First, it will increase the capital holdings of the capitalists, and increase the desire of the workers to set a high tax to extract from the capitalists. However, raising \( k \) for each capitalist will also increase the total amount of capital in society, and hence increase wages and decrease the rental rate of capital, \( r \). Assuming \( r_{\frac{1}{2}} > 0 \),

\[
\frac{\partial \nu'(r_{\frac{1}{2}})}{\partial k} = \frac{\partial \omega}{\partial k} [\bar{h} - h_{\frac{1}{2}}] - \lambda h \frac{\partial \omega}{\partial k} + \lambda r + \lambda k \frac{\partial r}{\partial k}
\]
\[ \frac{\partial v' (\tau_1^*)}{\partial k} = \{ \alpha \left[ (1-\lambda) \bar{h} \right]^{\alpha-1} (1-\alpha) (\lambda k)^{\alpha \lambda} \} \left[ (1-\lambda) \bar{h} - h_1 \right] + \lambda \{ (1-\alpha) (\lambda k)^{\alpha \lambda} \} \left[ (1-\lambda) \bar{h} \right]^{\alpha} \]

\[ \frac{\partial v' (\tau_2^*)}{\partial k} = \{ \alpha \left[ (1-\lambda) \bar{h} \right]^{\alpha} (1-\alpha) (\lambda k)^{\alpha \lambda} \} - \{ \alpha \left[ (1-\lambda) \bar{h} \right]^{\alpha-1} (1-\alpha) (\lambda k)^{\alpha \lambda} \} \left[ h_1 \right] + \lambda \{ (1-\alpha) (\lambda k)^{\alpha \lambda} \} \left[ (1-\lambda) \bar{h} \right]^{\alpha} \]

\[ \frac{\partial v' (\tau_1^*)}{\partial k} = \{ \lambda (1-\alpha) (\lambda k)^{\alpha} \} \left[ (1-\lambda) \bar{h} \right]^{\alpha} (1 - \frac{\alpha h_1}{(1-\lambda)h}) \]

\[ \frac{\partial v' (\tau_2^*)}{\partial k} = \{ \lambda r \} \left( 1 - \frac{\alpha h_1}{(1-\lambda)h} \right) \]

\[ \frac{\partial v' (\tau_3^*)}{\partial k} = \lambda r - (1-\alpha) \frac{\omega}{k} \left[ h_1 \right] \]

Hence we see that the effects upon the equilibrium tax rate of an increase in \( k \) are ambiguous. When \( \lambda \) and \( r \) are large, an increase in \( k \) will tend to increase the equilibrium tax rate, \( \tau_1^* \). (Remember, \( v'(\tau) \) is convex.) Intuitively, this is because when \( \lambda \) and \( r \) are large, capitalists are richer and there is more incentive to increase taxes to extract from them.

However when \((1-\alpha), \frac{\omega}{k}, \) and \( h_1 \) are large, an increase in \( k \) will lead to a decrease in the equilibrium tax rate. This arises because an increase in \( k \) leads to higher wages of the workers. As the median voter is a worker, the increase will raise his income, and lead him to desire lower taxes. The larger the share of income that goes to capital in society, the more human capital the median voter worker has, and the larger the ratio of wages to per capitalist capital, the more likely a rise in \( k \) will lead to a fall in the equilibrium tax.

Now we turn to how changes in the share of capitalists in society change the equilibrium tax rate. Intuitively there will be two affects. As \( \lambda \) increases, there are more capitalists, and the median voter becomes richer. This corresponds to a worker with more human capital and a desire for lower tax rates. However, as \( \lambda \) increases, each capitalist becomes poorer. This is due to the fact that the capitalists supply their output inelastically to the market; more capitalists means lower returns to capital \( r \). If \( \lambda \) becomes too large, capitalists will no longer be richer than the richest worker, and will move down the income distribution. Eventually, they will become poorer than the poorest worker and desire higher, not lower taxes. We compute this below. Again, \( \tau_1^* > 0 \).
\[ \omega \left[ \bar{h} - h_1 \right] + \lambda \left[ r k - \omega \bar{h} \right] = v'(\tau^*_1) \]

\[ \frac{\partial v'(\tau^*_1)}{\partial \lambda} = \frac{\partial \omega}{\partial \lambda} \left[ \bar{h} - h_1 \right] + \frac{\partial h_1}{\partial \lambda} \left[ r k - \omega \bar{h} \right] + \lambda k \frac{\partial r}{\partial \lambda} - \lambda \bar{h} \frac{\partial \omega}{\partial \lambda} \]

\[ \frac{\partial v'(\tau^*_1)}{\partial \lambda} = \frac{\partial \omega}{\partial \lambda} \left[ (1 - \lambda) \bar{h} \right] - f^1\left( \frac{1}{2(1 - \lambda)} \right) \frac{1}{2(1 - \lambda)^2} + \frac{\partial r}{\partial \lambda} \lambda k + \left[ r k - \omega \bar{h} \right] \]

**Aside:** \[ \omega = a[(1 - \lambda) \bar{h}]^{a-1}(\lambda k)^{1-a} = a[\bar{h}]^{a-1}(k)^{1-a}(1 - \lambda)^{1-a} = a[\bar{h}]^{a-1}(k)^{1-a}(\lambda^{-1})^{1-a} \]

\[ \frac{\partial \omega}{\partial \lambda} = \frac{(1 - \alpha)\omega}{\lambda(1 - \lambda)} \]

\[ r = (1 - \alpha)(\lambda k)^{1-a}(1 - \lambda) \bar{h}^a = (1 - \alpha)(k)^{1-a}(1 - \lambda)^a(\lambda^{-1})^a = (1 - \alpha)(k)^{1-a}(\bar{h})^a(1 - \lambda)^a(\lambda^{-1})^a \]

\[ \frac{\partial r}{\partial \lambda} = -\alpha r \]

\[ \frac{\partial v'(\tau^*_1)}{\partial \lambda} = \frac{(1 - \alpha)\omega}{\lambda(1 - \lambda)} \left[ (1 - \lambda) \bar{h} \right] - f^1\left( \frac{1}{2(1 - \lambda)} \right) \frac{1}{2(1 - \lambda)^2} + \frac{-\alpha r}{\lambda(1 - \lambda)} \lambda k + \left[ r k - \omega \bar{h} \right] \]

\[ \frac{\partial v'(\tau^*_1)}{\partial \lambda} = \frac{(1 - \alpha)\omega}{\lambda} - f^1\left( \frac{1}{2(1 - \lambda)} \right) \frac{(1 - \alpha)\omega}{2\lambda(1 - \lambda)^3} + \frac{-\alpha r}{(1 - \lambda)} + r k - \omega \bar{h} \]

\[ \frac{\partial v'(\tau^*_1)}{\partial \lambda} = \omega \bar{h} \left( 1 - \frac{(1 - \alpha)}{\lambda} \right) + r k \left( 1 - \frac{\alpha}{(1 - \lambda)} \right) - f^1\left( \frac{1}{2(1 - \lambda)} \right) \frac{(1 - \alpha)\omega}{2\lambda(1 - \lambda)^3} \]

Of the terms, the last one is the easiest to understand. It is always negative, and corresponds to the change in preferences of the median voter. When \( \lambda \) increases, there are more capitalists. As we have assumed so far that capitalists are richer than workers, then small increases in \( \lambda \) will preserve this and will correspond to an increase in the wealth of the median voter. A wealthier median voter will desire lower taxes, hence the final term enters as with a minus sign. The first and second terms are less intuitive. For any values of \( \alpha \) and \( \lambda \), either one term will be negative or both will be zero. This corresponds to the dynamics of how an increase in capital in the economy affects the wage and rental rates. (see discussion above)
For large increases in $\lambda$, capitalists will cease to be richer than workers. As long as $rk > \omega h_1$, this will not change the median voter, and will not change the equations above. However, it will correspond to some workers being poorer than a capitalist, and some workers with high human capital being richer than the capitalists. Once $rk < \lambda rk + (1-\lambda)\omega \overline{h}$, all capitalists begin to vote for a singular nonzero tax rate.

As $\lambda$ increases further, eventually a capitalist will become the median voter, and decide the equilibrium tax rate. Now, if $rk < \lambda rk + (1-\lambda)\omega \overline{h}$, this capitalist will set a $\tau > 0$ to extract from the richest workers, and be joined in voting for it by poorer workers. Finally $\lambda$ may increase so far as to make capitalists the poorest members of society. As such, they will not longer be the median voter, and the worker with $h_{1/2-\lambda}$ human capital will become the median voter. This worker will set the equilibrium tax rate at $\tau = 0$ if $\omega h_{1/2-\lambda} \geq \lambda rk + (1-\lambda)\omega \overline{h}$, and $v'(\tau_{1/2-\lambda}^*) = \omega[\overline{h} - h_{1/2-\lambda}] + \lambda[rk - \omega \overline{h}]$ if not.

B) The ideal tax rate for capitalists satisfies (don’t forget those boundary conditions)

$$\max_{\tau_k, \tau_h} rk(1 - \tau_k) + \tau_h[(1-\lambda)\omega \overline{h}] + \tau_k[\lambda rk] - v(\tau_h) - v(\tau_k)$$

$$\frac{\partial I}{\partial \tau_h} = (1-\lambda)\omega \overline{h} - v'(\tau_h) = 0$$

$$v'(\tau_{h,C}^*) = (1-\lambda)\omega \overline{h} \text{ for any } rk$$

and $\tau_{k,C}^* = 0$

Notice this is different from before. Capitalists would prefer to extract as much as possible from the workers, and will set $\tau_h$ to accomplish this, regardless of the capitalists’ income. Clearly they will also set $\tau_k = 0$, as all capitalists have equal income, and only lose resources by redistributing them to all of society through $\tau_k$.

The ideal tax rate for worker with human capital $h_i$ satisfies

$$\max_{\tau_k, \tau_h} \omega h_i(1 - \tau_h) + \tau_h[(1-\lambda)\omega \overline{h}] + \tau_k[\lambda rk] - v(\tau_h) - v(\tau_k)$$

$$\frac{\partial I}{\partial \tau_k} = \lambda rk - v'(\tau_k) = 0$$

$$v'(\tau_{k,C}^*) = \lambda rk$$

$$\frac{\partial I}{\partial \tau_h} = -\omega h_i + (1-\lambda)\omega \overline{h} - v'(\tau_h) = 0$$

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\[ v'(\tau^*_{h,i}) = \omega[\bar{h} - h_i] - \lambda\omega\bar{h} \]

if \( \omega h_i \geq (1-\lambda)\omega\bar{h} \Rightarrow \tau^*_{h,i} = 0 \)

if \( \omega h_i < (1-\lambda)\omega\bar{h} \Rightarrow \tau^*_{h,i} > 0 \)

If \( \lambda < \frac{1}{2} \), a voting equilibrium does not exist. Suppose that

\[ v'(\tau^*_{h,i}) = \lambda rk \text{ and} \]

\[ \tau_h = \tau^*_{h,\frac{1}{2}-\lambda}, \text{ where } v'(\tau^*_{h,\frac{1}{2}-\lambda}) = \omega[\bar{h} - h_{\frac{1}{2}-\lambda}] - \lambda\omega\bar{h} \]

was the equilibrium. A rival plan could be offered that would set

\[ \tau_k = \tau^*_{k} - \varepsilon \text{ and} \]

\[ \tau_h = \tau^*_{h,\frac{1}{2}-\lambda} - \Delta \]

\[ \varepsilon \geq \frac{[v'(\tau^*_{h,\frac{1}{2}-\lambda}) - (1-\lambda)\omega\bar{h}]}{\Delta} \]

and would succeed. All capitalist would vote for the plan, as it would lower their capital tax rate, while only slightly decreasing the revenue they receive from the human capital tax rate. Joining the capitalists, enough of the richest workers would vote for the change, as it would decrease their losses from the human capital taxation, while negligibly affecting their capital tax transfers (by the envelope theorem). But now a rival to this rival plan would offer

\[ v'(\tau^*_{k}) = \lambda rk \text{ and} \]

\[ \tau_h = \tau^*_{h,\frac{1}{2}-\lambda} - \Delta \]

and would succeed. All capitalists would vote against this plan, but would be outvoted by the \( 1-\lambda > \frac{1}{2} \) workers. Finally, this outcome would cycle back to the original,

\[ v'(\tau^*_{k}) = \lambda rk \text{ and} \]

\[ \tau_h = \tau^*_{h,\frac{1}{2}-\lambda} \]

as all capitalists and workers with \( h_i \leq h_{\frac{1}{2}-\lambda} \) would vote for higher human capital taxation.

Therefore, with the dual tax instrument, there is no voting equilibrium.

No voting equilibrium exists as long as \( 0 < \lambda < \frac{1}{2} \), however once \( \lambda \geq \frac{1}{2} \) a steady voting equilibrium exists. When capitalists exceed half of the population, one capitalist becomes the median voter. As such, his policy preferences are enacted. Further, because all capitalists share the same preferences, \( v'(\tau^*_{k,c}) = (1-\lambda)\omega\bar{h} \) and \( \tau^*_{k,c} = 0 \), workers can not split the vote of the capitalists and create a cycling between tax plans as above.

This case is different than with only the single tax instrument exactly because the dual instruments allow cycling between tax plans as seen above. Decreases in one tax instrument
can be exchanged with changes in the second, and this will be exploited to negate the possibility of a voting equilibrium. With a single tax instrument, this is not possible.

C) The workers and capitalist keep their ideal tax rates from part B). However, now with sequential voting, an equilibrium exists. The possibility of cycling between tax plans is eliminated because workers and capitalist can not contract on their vote in the second round of voting.

The first round of voting will result in all workers voting for \( v'(\tau^+_k) = \lambda r k \) and all capitalists voting against it. In the second round of voting, all capitalists and \( \frac{1}{1-\lambda} \) of the workers vote for the tax rate of \( \tau^+_{h, \frac{1}{2}-\lambda} \). As long as \( \omega \frac{h}{2-\lambda} < (1-\lambda) \omega \bar{h} \), \( \tau^+_{h, \frac{1}{2}-\lambda} \) will be nonzero. Hence in sequential voting, the equilibrium will be

\[
v'(\tau^+_k) = \lambda r k \quad \text{and} \quad \tau_h = \max(0, \tau^+_{h, \frac{1}{2}-\lambda})
\]

When \( k \) increases, the equilibrium tax rate changes as

\[
\frac{\partial v'(\tau^+_k)}{\partial k} = \lambda r + \lambda k \frac{\partial r}{\partial k} = \lambda r + \lambda k \{ (-\alpha)(1-\alpha)(\lambda k)^{a-1}\lambda \} \left[ (1-\lambda) \bar{h} \right]^a = \lambda r + \lambda k \frac{\alpha r}{k} = \lambda r - \alpha \lambda r = (1-\alpha) \lambda r
\]

\[
\begin{align*}
\frac{\partial v'(\tau^+_{h, \frac{1}{2}-\lambda})}{\partial k} &= \frac{\partial \omega}{\partial k} \left[ \frac{\bar{h} - h_{\frac{1}{2}-\lambda}}{2-\lambda} \right] - \frac{\lambda \bar{h}}{2-\lambda} \frac{\partial \omega}{\partial k} \\
&= \{ \alpha(1-\lambda)(\bar{h})^{a-1}(1-\alpha)(\lambda k)^{\alpha a}\lambda \} \left[ (1-\lambda) \bar{h} - h_{\frac{1}{2}-\lambda} \right] \\
&= \frac{(1-\alpha) \omega}{k} \left[ (1-\lambda) \bar{h} - h_{\frac{1}{2}-\lambda} \right] \text{if } \tau^+_{h, \frac{1}{2}-\lambda} \geq 0
\end{align*}
\]

When \( k \) increases,
\[
\frac{\partial v'(\tau_k^*)}{\partial k} = (1 - \alpha)\lambda
\]

\[
\frac{\partial v'(\tau_{h_{\frac{1}{2}-\lambda}}^*)}{\partial k} = \frac{(1 - \alpha)\omega}{k} \left[ (1 - \lambda) \bar{h} - h_{\frac{1}{2}-\lambda} \right] \text{ if } \tau_{h_{\frac{1}{2}-\lambda}}^* \geq 0
\]

\[
0 \quad \text{if } \tau_{h_{\frac{1}{2}-\lambda}}^* = 0
\]

That is, the tax rate on capital unambiguously rises, whereas the tax rate on human capital may rise, but does not necessarily. Neither tax rate may fall when \( k \) increases.

When \( \lambda \) increases, the equilibrium tax rate changes as

\[
\frac{\partial v'(\tau_k^*)}{\partial \lambda} = rk + \lambda k \cdot \frac{\partial r}{\partial \lambda}
\]

\[
= rk + \lambda k \cdot \frac{-\alpha r}{\lambda(1 - \lambda)}
\]

\[
= rk - \frac{\alpha kr}{(1 - \lambda)}
\]

\[
\frac{\partial v'(\tau_{h_{\frac{1}{2}-\lambda}}^*)}{\partial k} = \frac{\partial \omega}{\partial \lambda} \left[ (1 - \lambda) \bar{h} - \frac{\partial h_{\frac{1}{2}-\lambda}}{\partial \lambda} \right]
\]

\[
= \frac{(1 - \alpha)\omega}{\lambda(1 - \lambda)} \left[ (1 - \lambda) \bar{h} - f^1 \left( \frac{1 - 2\lambda}{2(1 - \lambda)} \right) \frac{-1}{2(1 - \lambda)^2} \right]
\]

\[
= \frac{(1 - \alpha)\omega \bar{h}}{\lambda} + f^1 \left( \frac{1 - 2\lambda}{2(1 - \lambda)} \right) \frac{(1 - \alpha)\omega}{2\lambda(1 - \lambda)^3}
\]

So for small \( \lambda \) increases,

\[
\frac{\partial v'(\tau_k^*)}{\partial \lambda} = rk - \frac{\alpha kr}{(1 - \lambda)}
\]

\[
\frac{\partial v'(\tau_{h_{\frac{1}{2}-\lambda}}^*)}{\partial k} = \frac{(1 - \alpha)\omega \bar{h}}{\lambda} + f^1 \left( \frac{1 - 2\lambda}{2(1 - \lambda)} \right) \frac{(1 - \alpha)\omega}{2\lambda(1 - \lambda)^3} \text{ if } \tau_{h_{\frac{1}{2}-\lambda}}^* \geq 0
\]

\[
0 \quad \text{if } \tau_{h_{\frac{1}{2}-\lambda}}^* = 0
\]

Here, the tax rate on capital may rise or fall, whereas the change in the tax rate on human capital must be nonnegative. The capital tax rate may rise for small changes in \( \lambda \) as when it is larger, there are more capitalists from which to extract taxes from. However, as \( \lambda \) rises, each capitalists become poorer because \( r \) falls. This will lead to a desire to decrease capital taxes because of the dead weight loss to taxation.
The tax rate on human capital may rise for two reasons. More capitalists means that the wage rate will rise, and that will cause the incentive to tax human capital to rise. Also, with more capitalists, the capitalists need fewer workers to attain a majority of voters. As a result of this, the median voter will be a worker with a lower level of human capital, and hence a stronger desire for a higher tax rate. However once \( \lambda \) passes \( \lambda > \frac{1}{2} \) a capitalist becomes the median voter. In this case, his policy preferences become the voting equilibrium outcome. As all capitalists are identical there are no further changes in \( \tau^*_w, \tau^*_h \) for increases in \( \lambda \) after it passes \( \frac{1}{2} \). The equilibrium tax rates for \( \lambda > \frac{1}{2} \) are

\[
v'(\tau^*_k) = (1-\lambda)\omega \tilde{H} \quad \text{for any } rk
\]

\[
\text{and } \tau^*_k = 0
\]