A Practical Introduction to
Differential Forms

William C. Schulz

and

Alexia E. Schulz

September 8, 2012

Transgalactic Publishing Company
Flagstaff, Vienna, Cosmopolis
Contents

1 Introduction and Basic Applications 1
   1.1 INTRODUCTION ........................................ 2
   1.2 Some Conventions ...................................... 3
   1.3 Some Formulas to Recall ............................... 4
   1.4 Coordinate systems .................................... 6
   1.5 The Algebra of Differential Forms ................... 6
   1.6 The Operator $d$ .................................... 8
   1.7 Orientation ........................................... 9
   1.8 Differential Forms and Vectors ..................... 11
   1.9 grad, curl and div .................................... 12
   1.10 The Poincaré Lemma and it’s Converse ............. 13
   1.11 Boundaries .......................................... 16
   1.12 Integrals of Forms .................................. 18
   1.13 Variable Changes .................................... 19
   1.14 Surface integrals ................................... 20
   1.15 The Generalized Stokes Theorem .................... 22
   1.16 Curvilinear Coordinates I: preliminary formulas .. 25
   1.17 Curvilinear Coordinates II: the calculations ...... 29
   1.18 Surfaces and Manifolds .............................. 36
   1.19 The Dualizing Operator $\ast$ ......................... 36
   1.20 The Codifferential $\delta$ ............................ 43
   1.21 The Laplacian ....................................... 45
   1.22 Maxwell’s Equations in 3-space ..................... 46

2 Mathematical Theory 51
   2.1 INTRODUCTION ........................................ 52
   2.2 Permutations .......................................... 52
   2.3 The operator $\Phi$ ................................... 53
Chapter 1

Introduction and Basic Applications
1.1 INTRODUCTION

These notes began life as an introduction to differential forms for a mathematical physics class and they still retain some of that flavor. Thus the material is introduced in a rather formal and manner and the mathematical complexities are put off to later sections. We have tried to write so that those whose primary interest is in the applications of differential forms can avoid the theoretical material provided they are willing to accept the formulas that are derived in the mathematical sections, which are clearly marked as such. Those who wish may read the mathematical sections as they occur, or later, or indeed may put them off to a more convenient time, perhaps in a future life, without loss to the continuity of the applied thread. Anyway, such is my hope. But we want to also emphasize that those who wish will find all the mathematical details available, at a level of rigor usual to the better mathematical physics books. The treatment is mostly local, and what little manifold theory is needed is quietly developed as we go. We have tried to introduce abstract material in circumstances where it is useful to do so and we have also tried to avoid introducing a lot of abstract mathematical material all at once time.

The two areas most completely addressed in these notes, besides the foundational material, are coordinate changes and Maxwell’s equations since we feel that these illustrate the power of differential forms quite well. We treat Maxwell’s equations in both three and four dimensions in separate sections. We will also look at a few other things.

Notation has been carefully chosen to be consistent with standard tensor notation to facilitate comparison with such treatments, and to facilitate learning basic differential geometry.

The treatment of Maxwell’s equations requires the derivation of the potential equations. Although not strictly necessary, we have introduced the codifferential δ and the Laplace operator ∇d + δd since this is the natural route using modern mathematics. For example we point out that the condition of Lorenz can be expressed instantly and easily in terms of the codifferential in four dimensions. And as long as we have it available we can look at a couple of other applications of the Laplace operator on forms.

A justified criticism of these notes might be that many things are done twice, which is not efficient. We have sacrificed efficiency for convenience to the reader who may wish to deal with only one particular thing, and so would like a relatively complete treatment in the section without having to read five others. Similarly, many formulas are repeated at the beginning of sections where they are used, rather than referred to in previous sections. The increase in paper is rather small, and for those getting it electronically there is no waste at all. It is difficult for a mathematician to resist the call of generality but since one of us is a physicist the brakes have been applied, and we hope that the product is a reasonable compromise between the siren song of mathematics and the needs of practical physics.
1.2 Some Conventions

Here we will introduce some conventions that will be used throughout these notes. The letter $A$ will be used for a region of 2-dimensional space, for example the unit disk consisting of points whose distance from the origin is less than or equal to 1. Its boundary would be the unit circle consisting of points whose distance from the origin is exactly 1. We will use the symbol $\partial$ to indicate the boundary. Thus if $A$ is the unit disk $A = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$ then the boundary of $A$ is $\partial A = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ which is the unit circle. Notice carefully the difference between the terms DISK and CIRCLE. (DISK and CIRCLE are often confused in common speech.)

The letter $M$ will be used for a (solid) region of 3-dimensional space, for example the unit ball, $M = \{x \in \mathbb{R}^3 \mid |x| \leq 1\}$ whose boundary is the unit sphere $\partial M = \{x \in \mathbb{R}^3 \mid |x| = 1\}$. (The terms BALL and SPHERE are often confused in common speech.)

The letter $S$ will be used for a (2-dimensional) surface in three dimensional space, for example the upper half of the unit sphere. The boundary of this $S$ would be a circle in the $x, y$ plane.

If we do not wish to specify dimension, we will use the letter $K$. The use of $K$ indicates that the formula will work in any dimension, and this usually means any dimension, not just 1, 2 or 3 dimensional space. Naturally $\partial K$ means the boundary of $K$.

The ball and sphere have analogs in every dimension. It is customary to refer to the ball in $\mathbb{R}^n$ as the $n$-ball and its boundary as the $(n-1)$-sphere. For example, the unit disk is the 2-ball and its boundary, the unit circle, is the 1-sphere. Note that the $m$-sphere lives in $\mathbb{R}^{m+1}$. It is called the $m$-sphere because it requires $m$ variables to describe it, like latitude and longitude on the 2-sphere.

Also useful to know are the terms open and closed. This is a tricky topological concept, so we will treat it only intuitively. $K$ is closed if it includes its boundary. Thus the unit disk and unit ball are closed. If we remove the boundary $\partial K$ from $K$ the resulting set $K^\circ$ is called open. Thus for the unit ball in $\mathbb{R}^3$ we have

\[
\begin{align*}
M &= \{x \in \mathbb{R}^3 \mid |x| \leq 1\} \quad \text{closed ball} \\
M^\circ &= \{x \in \mathbb{R}^3 \mid |x| < 1\} \quad \text{open ball} \\
\partial M &= \{x \in \mathbb{R}^3 \mid |x| = 1\} \quad \text{sphere}
\end{align*}
\]

We want to give a real world example here but remember it must be inexact since real world objects are granular (atomic) in constitution, so can only approximate the perfect mathematical objects. Some people prefer to eat the closed peach (with fuzzy skin), some people prefer the open peach (fuzzy skin removed, peach$^\circ$) and the boundary of the peach, $\partial$peach, is the fuzzy skin. Perhaps this will help you remember. Deeper knowledge of these matters can be found in the wonderful book [2] and also [3].
For functions we will use a slightly augmented variant of the physics convention. When we write \( f : S \to \mathbb{R} \) we mean a function whose input is a point \( p \in S \) and whose output is a real number. This is theoretically useful but not suitable for calculation. When we wish to calculate, we need to introduce coordinates. If we are dealing with the upper half of the unit sphere (set of points in \( \mathbb{R}^3 \) whose distance from the origin is exactly one and for which \( z \geq 0 \)) then we might write \( f(x,y) \) if we choose to represent points in the \( x,y \) coordinate system. Notice, and this is an important point, that the coordinate \( x \) takes as input \( p \in S \) and outputs a real number, it’s \( x \) coordinate. Hence the coordinates \( x \) and \( y \) are functions just like \( f \). If \( S \) is the upper half of the unit sphere in \( \mathbb{R}^3 \) then \( x \) and \( y \) are not really good coordinates. It would be be better to use longitude and colatitude for my coordinates and then we would write \( f(\phi, \theta) \). ¹ Note use of the same letter \( f \) no matter what the coordinate system, because the \( f \) represents a quantity in physics, whereas in math it represents a functional relationship and we would not use the same letter for different coordinates. Note also that \( f(\pi, \pi) \) is ambiguous in physics unless you have already specified the coordinate system. Not so with the math convention.

Finally, we will almost always use the letters \( f, g, h \) for functions on \( A, M, S, K \). Mostly these will occur in coordinate form, for example \( f(x, y, z) \) for a function on \( M \).

### 1.3 Some Formulas to Recall

You are all familiar with the \( dx, dy, dz \) which occur in the derivative notation \( \frac{dv}{dx} \) and the integral notation

\[
\int_M f(x, y) \, dxdy
\]

\[
\int_M f(x, y, z) \, dxdydz
\]

and you recall the Green, divergence and Stokes theorems.

\[
\int_{\partial A} f(x, y) \, dx + g(x, y) \, dy = \int_A \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \, dxdy
\]

\[
\int_{\partial M} f(x, y, z) \, dydz + g(x, y, z) \, dzdx + h(x, y, z) \, dxdy = \int_M \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \, dxdydz
\]

\[
\int_{\partial S} f(x, y, z) \, dx + g(x, y, z) \, dy + h(x, y, z) \, dz = \int_M \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \, dydz + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \, dzdx + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dxdy
\]

¹BEWARE. \( \phi \) is longitude in physics but colatitude in mathematics. \( \theta \) is colatitude in physics but longitude in math.
1.3. SOME FORMULAS TO RECALL

You might be more familiar with the last two in the vector forms

$$\int_{\partial M} \mathbf{v} \cdot d\mathbf{S} = \int_M \text{div} \, \mathbf{v} \, dV$$

and

$$\int_{\partial S} \mathbf{v} \cdot d\mathbf{\ell} = \int_S \text{curl} \, \mathbf{v} \cdot d\mathbf{S}$$

There are some conventions on integrals that we will mention now. In former times when integrating over a three dimensional object we would write $\int \int \int_M \text{div} \, \mathbf{v} \, dV$. This is now completely antiquated, and we will not do it.

On the other hand, there is a convention that when integrating around curves or surfaces that have no boundary we put a small circle on the integral, so that we write

$$\oint_{\partial M} \mathbf{v} \cdot d\mathbf{S}$$

for the integral $\int_{\partial M} \mathbf{v} \cdot d\mathbf{S}$. Since this is favored by the physics community we will mostly use it. Notice that if a geometric object is the boundary of something, then it itself has no boundary, and so we will use the circled integral almost exclusively with boundaries.

For our purposes we will define a differential form to be an object like

$$f(x,y) \, dx \quad f(x,y,z) \, dydz \quad f(x,y,z) \, dxdydz$$

which we find as integrands in the written out forms of the Green, divergence and Stokes theorem above. If $\omega$ is a sum of such objects it turns out that the three theorems collapse to one mighty theorem, called the generalized Stokes theorem, which is valid for all dimensions:

$$\oint_{\partial S} \omega = \int_S d\omega$$

To use this theorem and for other purposes it is only necessary to

1. Learn the algebra that the $dx, dy, dz$ satisfy which is almost the same as ordinary algebra with one important exception.

2. Learn the rule for the operator $d$ which is almost trivial.

Once these are learned differential forms can be manipulated easily and with confidence. It is also useful to learn how various things that happen in vector analysis can be mimicked by differential forms, and we will do this, naively at first and then in much more detail.

If you are concerned about what differential forms ARE, the answer is a little tricky and we are going to put it off for the moment. Later we will discuss the surprisingly dull answer to this question. Incidentally, the difficulty in explaining what they really are is one reason they have not become more common in elementary textbooks despite their extreme usefulness.

Just to give a tiny hint of the geometrical interpretation of differential forms. A two form measures the density of lines of force of a field, as introduced by James Faraday a century and a half ago. For more on this subject see [1] or [5]. We will discuss it a bit more when we have more equipment.
1.4 Coordinate systems

Our first job is to talk a little about coordinates. You already know most of this so we can do it quickly. As already specified, we will use the notation $A$ for a finite region in $\mathbb{R}^2$ (which is the standard notation for ordinary two dimensional real space). We will use $M$ for a three dimensional finite region of $\mathbb{R}^3$ and $S$ for a curved surface in $\mathbb{R}^3$. Standard coordinates for $A$ would be $x, y$, but we might want to use polar coordinates $r, \theta$ or even more general coordinates $u, v$. The important thing for $A$ is that there be two coordinates and that they be "independent", so that $v$ is not a function of $u$.

In the case of $M$, a three dimensional region of $\mathbb{R}^3$, we will need three coordinates $x, y, z$ or $r, \theta, \phi$ or more generally $u, v, w$ to describe it.

Since $S$ is a surface in $\mathbb{R}^3$, it will be described by two coordinates. In elementary cases this will often be $x, y$ but in more complicated situations it is often wise to taylor$^2$ the coordinates to the geometric object, for example if one is working with the Unit Sphere in $\mathbb{R}^3$ then the appropriate coordinates would be $\theta, \phi$ (longitude and colatitude). Proper choice of coordinates can make a nasty problem much more pleasant.

It is important to be able to move from one coordinate system for a geometric situation to a different coordinate system and we will discuss this later.

1.5 The Algebra of Differential Forms

We now begin our discussion of the algebra of differential forms. The multiplication symbol used is $\wedge$ as in $dx \wedge dy$ but this is very often omitted. We will use it for a while, and then omit it when we get bored with it.

Let us start with a function in $f(x, y, z)$ on $\mathbb{R}^3$. You already know how to form $df$:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Recalling that, like $f$, the coordinate $x$ is also a function on $\mathbb{R}^3$ the previous formula writes the differential of $f$ in terms of the differentials of the three special functions $x, y, z$. So we note that there is not much difference between $df$ and $dx$; they are the same kind of object. All objects of this type (differentials of functions) are collected together in the set

$$\Lambda^1(\mathbb{R}^3)$$

and are called 1-forms.

If we multiply a couple of these together we get objects like

$$f \, dx \wedge g \, dy = fg \, dx \wedge dy \in \Lambda^2(\mathbb{R}^3)$$

$^2$pun
Note that functions \( f \) commute with \( dx \): \( f \, dx = dx \, f \); see below. Linear combinations of such objects are called 2-forms. And of course there are 3-forms

\[
\begin{align*}
f \, dx \wedge dy \wedge dz & \in \Lambda^3(\mathbb{R}^3)
\end{align*}
\]

To complete the system we will place the functions in the basement of the building: \( f \in \Lambda^0(\mathbb{R}^3) \). It is customary to omit the wedge when multiplying by a function; we write

\[
\begin{align*}
f \, dx \wedge dy & \quad \text{for} \quad f \wedge dx \wedge dy
\end{align*}
\]

There is no significance to this; it is just convention.

The algebra of these objects is just like ordinary algebra except for the changes caused by the rule

\[
\begin{align*}
dg \wedge df = - df \wedge dg & \quad \text{(anti-commutativity)}
\end{align*}
\]

for the 1-forms \( df \) and \( dg \). An algebra satisfying this rule is called an exterior or Grassmann algebra. This algebra was invented by Hermann Graeventh Grassmann about 1840 in an attempt to find an algebra mirroring elementary geometry. It is sufficient to postulate this rule for the coordinate differentials only,

\[
\begin{align*}
\begin{align*}
dy \wedge dx & = - dx \wedge dy \\
\end{align*}
\end{align*}
\]

since the general rule will follow by linearity.

Thus the exterior algebra is not commutative. Our rule is often called anti-commutative and is the simplest generalization of commutative, but it has many consequences which seem strange to the beginner. For example, if we substitute \( f \) for \( g \) in the rule \( dg \wedge df = - df \wedge dg \) we get

\[
\begin{align*}
\begin{align*}
df \wedge df & = - df \wedge df
\end{align*}
\end{align*}
\]

so moving the right side to the left side by the usual algebraic processes which are all valid here we get

\[
\begin{align*}
\begin{align*}
df \wedge df + df \wedge df & = 0 \\
2 df \wedge df & = 0 \\
df \wedge df & = 0
\end{align*}
\end{align*}
\]

Thus the product of a one form with itself is 0, which is very important. Let’s look at another example

\[
\begin{align*}
\begin{align*}
(f \, dx + g \, dy) \wedge (f \, dx + g \, dy) & = f \, f dx \wedge dx + f \, g \, dx \wedge dy + g \, f \, dy \wedge dx + g \, g \, dy \wedge dy \\
& = 0 + fg(dx \wedge dy + dy \wedge dx) + 0 \\
& = fg0 = 0
\end{align*}
\end{align*}
\]

as promised.
We also see from this that there are no four forms in 3-space, since if we multiply four of the objects $dx, dy, dz$ together there will be a repetition which will kill the form:

$$dx \wedge dy \wedge dz \wedge dx = - dx \wedge dy \wedge dx \wedge dz = dx \wedge dx \wedge dy \wedge dz = 0 \wedge dy \wedge dz = 0$$

In general, for a space $K$ of dimension $n$ there will be forms $\omega \in \Lambda^j(K)$ for $j = 0, \ldots, n$. It is not true in general that for $j$-forms $\omega$ with $j \geq 2$ that $\omega \wedge \omega = 0$ although this is fortuitously true for dimensions $n \leq 3$. There is an example in dimension 4 where $\omega \wedge \omega \neq 0$ in the problems.

Now let us start to deal a little more abstractly, so we will use a region $K$ of dimension $n$, and consider $\omega \in \Lambda^j(K)$ and $\eta \in \Lambda^k(K)$. Then a little thought shows that

$$\eta \wedge \omega = (-1)^{jk} \omega \wedge \eta$$

For example, with $\omega = dx \in \Lambda^1(\mathbb{R}^3)$ and $\eta = dy \wedge dz \in \Lambda^2(\mathbb{R}^3)$ we have

$$(dy \wedge dz) \wedge dx = dy \wedge dz \wedge dx$$

$$= - dy \wedge dx \wedge dz$$

$$= dx \wedge dy \wedge dz$$

and if you look at how this special case works you will see why the general case works.

Note that nothing prevents us from mixing things up as in

$$2dx + 3dx \wedge dy$$

but such things do not occur in practice. Forms where each term has the same number of differentials (forms of the same degree) are called homogeneous, and we almost always use homogeneous expressions.

### 1.6 The Operator $d$

Our next project is the differential operator $d$, which we introduce according to the following, where $K$ is a space with coordinates $x^1, \ldots, x^n$:

- $d$ is the unique operator that satisfies the following laws
  
  1. $d$ is a linear operator
  2. On the functions $\Lambda^0(K)$, $df$ is given by the the advanced calculus formula
     $$df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n$$
  3. if $\omega \in \Lambda^j(K)$ and $\eta \in \Lambda^k(K)$ then (Leibniz’s Rule)
     $$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^j \omega \wedge d\eta$$
1.7. **ORIENTATION**

4. $dd = 0$

Let’s look at some examples in 3-space of how these rules make everything work. First we examine $d(fdx)$. Since $f \in \Lambda^0(\mathbb{R}^3)$ we have, using rule 2,

$$d(fdx) = df \wedge dx + (-1)^0 f \wedge ddx = df \wedge dx + 0$$

we also used rule 3, $ddx = 0$, in the last equation. This derives the practical rule in a special case, and the general case (see problems) will be the same.

$$d(fdx^{i_1} \wedge \ldots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

This is the practical rule for $d$ and the one you will use for almost everything you do, so learn it well.

Now let $\omega = fdx$ and $\eta = gdy$. Then we have

$$d(fdx \wedge gdy) = d(fdx) \wedge gdy + (-1)^1 fdx \wedge d(gdy)$$
$$= (df \wedge dx) \wedge gdy - fdx \wedge (dg \wedge dy)$$
$$= df \wedge dx \wedge dy + fdy \wedge dx \wedge dy$$
$$= (df + fdg) \wedge dx \wedge dy$$
$$= d(fdg) dx \wedge dy$$

just as we expected from the practical rule since $fdx \wedge gdy = fdg dx \wedge dy$. This also illustrates how rule 2 is a generalization of Leibniz’s formula for the derivative of a product.

### 1.7 Orientation

At this point we must deal with one of the less pleasant aspects of elementary geometry which is orientation. An orientation, to speak loosely, is a sense of twist in a space. For example, in $\mathbb{R}^2$ our standard sense of twist is counterclockwise; we measure angles from the $x$ axis in the direction of the $y$ axis. If we reverse either axis, we get the opposite twist. If the $y$ axis went *down* instead of *up* then we would measure angles in a clockwise direction and $\mathbb{R}^2$ would have the *opposite* orientation. If we reverse both axes then the sense of twist returns to counterclockwise.

In $\mathbb{R}^3$ the standard orientation is given by the following rule: if you place the fingers of your **RIGHT** hand so that the fingers curl from the $x$ to the $y$ coordinate axes then your thumb points in the direction of the $z$ axis. This is called the **right hand rule**. It has become standard to use this in Calculus books only since the 1940’s, and the opposite convention is still found in Italian books, so it is wise to check. To appreciate the subtlety of this concept, think of trying to communicate it to inhabitants of a planet in another galaxy. Since nature is almost symmetric in terms of left and right, the only way we know to clarify this is certain non-symmetric aspects of beta decay. This is referred to
in physics as parity. Hopefully parity properties remain the same from galaxy to galaxy.

This idea of orientation, though subtle, is strongly coupled with differential forms and is the reason for the anticommutativity. It also is a prime source of mistakes, and great care must be taken to keep things in correct order. In $\mathbb{R}^2$ (and its subspaces) the correct order, which expresses the orientation properly, is $dx \wedge dy$. Thus $dy \wedge dx$ is in incorrect order as indicated by the minus sign in

$$dy \wedge dx = -dx \wedge dy$$

Recall that when using Green’s theorem

$$\oint_{\partial A} f(x, y) \, dx + g(x, y) \, dy = \int_A \frac{\partial g}{\partial y} - \frac{\partial f}{\partial x} \, dxdy$$

it is critical that the direction around the boundary of the left integral be counterclockwise. If it is taken clockwise then the two sides of the equation will have opposite signs. This is again due to the orientation which is built into $\mathbb{R}^2$ but which we seldom notice explicitly. There are similar worries in the use of the divergence theorem and Stokes theorem.

In applications, the principal place where orientation occurs in $\mathbb{R}^n$ is in $n$-forms and $(n-1)$-forms. We will first tell you the general formula and then give you practical methods to make orientation (relatively) easy to deal with. Let the variables be used in $\mathbb{R}^n$ be $u^1, u^2, \ldots, u^n$. The use of superscripts to number the variables is to conform to tensor analysis standards and we don’t need to go into the reasons for it here; just do it! And remember $u^3$ is the third variable, not the cube of $u$. If we choose an order for the variables, which we did by numbering them, this chooses one of the two orientations. Then

$$du^1 \wedge du^2 \wedge \ldots \wedge du^n$$

is in correct order

$$du^2 \wedge du^1 \wedge \ldots \wedge du^n$$

is in incorrect order

because

$$du^2 \wedge du^1 \wedge \ldots \wedge du^n = -du^1 \wedge du^2 \wedge \ldots \wedge du^n$$

As you can see with a little practice, interchanging any two of the $du^i$ reverses the sign and changes correct to incorrect order or incorrect to correct order. If you are familiar with permutations, odd permutations of $du^1 \wedge du^2 \wedge \ldots \wedge du^n$ give incorrect order and even permutations give correct order.

That part is easy. The tricky part is the $(n-1)$-forms. Here the correct order is (with $du^i$ missing from the list)

$$(-1)^{i-1}du^1 \wedge \ldots \wedge du^{i-1} \wedge du^{i+1} \wedge \ldots \wedge du^n$$

The reason for this choice is so that

$$du^i \wedge ((-1)^{i-1}du^1 \wedge \ldots \wedge du^{i-1} \wedge du^{i+1}) = du^1 \wedge \ldots \wedge du^n$$
which is correct because the $du^i$ must hop over the $n-1$ elements $du^1, \ldots, du^{i-1}$ in order to get back into correct order and each hop contributes a minus sign.

So much for theory. In $\mathbb{R}^3$ correct order is

$$dx \wedge dy \wedge dz$$

and for 2-forms we have

$$dy \wedge dz, \quad -dx \wedge dz, \quad dx \wedge dy$$

For practical use it is much better to write these in cyclic order.\(^3\)

$$dy \wedge dz, \quad dz \wedge dx, \quad dx \wedge dy$$

and the correct order can be easily remembered by writing

$$dx dy dz dx dy dz$$

and noting that the order of a wedge of two differentials is correct if it occurs in this list, for example $dz \wedge dx$ is correct but $dx \wedge dz$ is incorrect since $dx dz$ does not occur in the list. Other incorrects are $dy \wedge dx$ and $dz \wedge dy$. The use of differential forms in $\mathbb{R}^3$ relies critically on writing things with correct order.

### 1.8 Differential Forms and Vectors

Standard 3-dimensional vector analysis was cobbled together by Josiah Willard Gibbs in the 1890s using pieces from a variety of sources. While it works well for many practical purposes it has large deficiencies from a theoretical standpoint. Those parts of it which concern the dot (scalar) product are fine, but those parts which concern the cross (vector product $\mathbf{v} \times \mathbf{w}$) are mathematically clumsy. To see this, consult any textbook for a proof of the vector triple product law

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

It is often said that the cross product cannot be generalized to higher dimensions but this is not true; what is true is that the analog of the cross product in $n$ dimensions involves not two but $n-1$ vectors. Thus the elementary geometric applications of the cross product can often be reproduced, but the physical applications not so much, which is the reason that for relativity (4 dimensions and space-time rather than just space) we must fall back on Tensor Analysis.

In 3 dimensions there are many formal analogies between differential forms and vector analysis. That is, differential forms will act like vectors in many ways. It is a little difficult to find good mathematical reasons for these analogies and we will not concern ourselves here with what these reasons might be, although we will return to the matter later. The practical consideration is that things

\(^3\)Cyclic order is a 3-space concept and does not generalize to n-space at all well.