Online appendix for Conditioning and Updating under Cumulative Prospect Theory

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Full proof of Theorem 1.

**Sufficiency:** We first prove that $f_\ast \in F_\ast$ is sufficient for the decomposition. First consider conditioning on $s \in A$. Let $\alpha$ be a function defined on $\{1, \ldots, n-k\}$ that orders the states in $A^\ast$ such that $f_\ast(\alpha(n-k)) \succeq \cdots \succeq f_\ast(\alpha(1))$. With the benchmark prospect $f_\ast \in F_\ast$, we have

$$V(h_Af_\ast) = V(h_\ast 1, \ldots, h_\ast k, f_\ast(\alpha(1)), \ldots, f_\ast(\alpha(n-k)))$$

$$= \sum_{i=1}^{k} \left( w^- \left( \sum_{j=1}^{i} p_{\alpha(j)} \right) - w^- \left( \sum_{j=1}^{i-1} p_{\alpha(j)} \right) \right) v(h_\ast i)$$

$$+ \sum_{i=1}^{n-k} \left( w^+ \left( \sum_{j=i}^{n-k} p_{\alpha(j)} \right) - w^+ \left( \sum_{j=i+1}^{n-k} p_{\alpha(j)} \right) \right) v(f_\ast(\alpha(i))).$$

and

$$V(g_Af_\ast) = V(g_\ast 1, \ldots, g_\ast k, f_\ast(\alpha(1)), \ldots, f_\ast(\alpha(n-k)))$$

$$= \sum_{i=1}^{k} \left( w^- \left( \sum_{j=1}^{i} p_{\alpha(j)} \right) - w^- \left( \sum_{j=1}^{i-1} p_{\alpha(j)} \right) \right) v(g_\ast i)$$

$$+ \sum_{i=1}^{n-k} \left( w^+ \left( \sum_{j=i}^{n-k} p_{\alpha(j)} \right) - w^+ \left( \sum_{j=i+1}^{n-k} p_{\alpha(j)} \right) \right) v(f_\ast(\alpha(i))).$$

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Thus,

\[ h_{Af_s} \succsim g_{Af_s} \]

\[ \iff \quad V(h_{Af_s}) \geq V(g_{Af_s}) \]

\[ \equiv \sum_{i=1}^{k} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(h_{o(i)}) + \sum_{i=1}^{n-k} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(f_s(\alpha(i))) \]

\[ \geq \sum_{i=1}^{k} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(g_{o(i)}) + \sum_{i=1}^{n-k} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(f_s(\alpha(i))) \]

so preferences conditional on the information that \( s \in A \) are represented by \( E^f_s \{ h \mid s \in A \} \).

Now consider conditioning on the complement \( A^c = \{ o(k+1), \ldots, o(n) \} \). We have

\[ V(h_{Ac}f_s) = V(x_s, \ldots, x_s, h_{o(k+1)}, \ldots, h_{o(n)}) \]

\[ = w^{-\left( \sum_{j=1}^{k} P_{o(j)} \right)} v(x_s) + \sum_{i=k+1}^{n} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(h_{o(i)}) \]

and

\[ V(g_{Ac}f_s) = V(x_s, \ldots, x_s, g_{o(k+1)}, \ldots, g_{o(n)}) \]

\[ = w^{-\left( \sum_{j=1}^{k} P_{o(j)} \right)} v(x_s) + \sum_{i=k+1}^{n} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(g_{o(i)}) \]

Thus,

\[ h_{Ac}f_s \succsim g_{Ac}f_s \]

\[ \iff \quad V(h_{Ac}f_s) \geq V(g_{Ac}f_s) \]

\[ \equiv \sum_{i=k+1}^{n} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(h_{o(i)}) \geq \sum_{i=k+1}^{n} \left( w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)} \right) v(g_{o(i)}) \]

\[ \equiv \sum_{i=k+1}^{n} \frac{w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)}}{1 - w^{-\left( \sum_{j=1}^{k} P_{o(j)} \right)}} v(h_{o(i)}) \geq \sum_{i=k+1}^{n} \frac{w^{-\left( \sum_{j=1}^{i} P_{o(j)} \right)} - w^{-\left( \sum_{j=1}^{i-1} P_{o(j)} \right)}}{1 - w^{-\left( \sum_{j=1}^{k} P_{o(j)} \right)}} v(g_{o(i)}) \]
so preferences conditional on the information that \( s \in A^c \) are represented by \( E^m_L[h|s \in A^c] \).

Since \( p_A = \sum_{j=1}^k p_{o(j)} \), it now follows that in the loss domain

\[
\begin{align*}
&w^-(p_A) E^r_L[h|s \in A] + (1 - w^-(p_A)) E^m_L[h|s \in A^c] \\
= &\quad w^- \left( \sum_{j=1}^k p_{o(j)} \right) \sum_{i=1}^k w^- \left( \sum_{j=1}^i p_{o(j)} \right) - w^- \left( \sum_{j=1}^{i-1} p_{o(j)} \right) v(h_{o(i)}) \\
&\quad + (1 - w^- \left( \sum_{j=1}^k p_{o(j)} \right)) \sum_{i=k+1}^n w^- \left( \sum_{j=1}^i p_{o(j)} \right) - w^- \left( \sum_{j=1}^{i-1} p_{o(j)} \right) v(h_{o(i)}) \\
= &\quad \sum_{i=1}^n \left( w^- \left( \frac{i}{k} \sum_{j=1}^k p_{o(j)} \right) - w^- \left( \sum_{j=1}^{i-1} p_{o(j)} \right) \right) v(h_{o(i)}),
\end{align*}
\]

which equals the unconditional CPT utility. Hence, we have the desired decomposition.

\textbf{Necessity:} We now turn to showing necessity. We proceed by showing that if \( f \notin F_* \), then we can choose \( w^-(\cdot) \) and \( v(\cdot) \) such that conditional preferences are not represented as stated in part (a) of the theorem. That is, we can find functions \( w^-(\cdot) \) and \( v(\cdot) \) satisfying the stated conditions such that \( g_{A^c} f > h_{A^c} f \) but \( E^m_L[h|s \in A^c] \geq E^m_L[g|s \in A^c] \), or \( g_A f > h_A f \) but \( E^r_L[h|s \in A] \geq E^r_L[g|s \in A] \), for some acts \( h, g \in \mathcal{H}_L \). A prospect \( f \notin F_* \) if \( f_s \succ x_s \) for some \( s \in A \) or \( 0 \succ f_s \) for some \( s \in A^c \).

We first show that if \( f_s \succ x_s \) for some \( s \in A \), then we can have \( g_{A^c} f > h_{A^c} f \) but \( E^m_L[h|s \in A^c] \geq E^m_L[g|s \in A^c] \). So suppose that there indeed exists \( s \in A \) such that \( f_s \succ x_s \). Since there are at least four outcomes in the loss domain and at least two states in \( A^c \), we can choose prospects \( h, g \in \mathcal{H}_L \) such that

\[
0 \succ h_{o(k+2)} \succ g_{o(k+2)} \succ g_{o(k+1)} \succ h_{o(k+1)} \succ x_s
\]

and \( h_{o(i)} = g_{o(i)} = h_{o(k+2)} \) for all \( i > k + 2 \). Depending on \( f \), there are different cases we need to consider.

The different cases arise because the details of the proof depend on how \( f \) relates to the outcomes in the loss domain (recall that all we require is that there are at least four outcomes in \( X_L \)). When there are more than four outcomes in \( X_L \), we may simultaneously be able to use the approach of two different cases. The procedure is the same in each case: We calculate the utility of \( h_{A^c} f \) and of \( g_{A^c} f \) and derive conditions for \( g_{A^c} f > h_{A^c} f \) and for \( E^m_L[h|s \in A^c] \geq E^m_L[g|s \in A^c] \). We then show that there exist \( w^-(\cdot) \) and \( v(\cdot) \) such that both of these conditions are satisfied. Since such \( w^-(\cdot) \) and \( v(\cdot) \) exist, preferences are not represented as in statement (a) of Theorem 1.

To this end, we introduce the following notation: Let \( \hat{A} = \{ s \in A | f_s \succ x_s \} \) and let \( \hat{A}_L = \{ s \in \hat{A} | 0 \succ f_s \} \). Note that \( \hat{A} \) is non-empty by the supposition above. Let \( |\hat{A}| \) and \( |\hat{A}_L| \) denote the number of states in \( \hat{A} \) and \( \hat{A}_L \), respectively, and let \( \tau \) be a function defined on \( \{1, \ldots, |\hat{A}|\} \) that orders the states in \( \hat{A} \) such that \( f_{\tau(|\hat{A}|)} \succ \cdots \succ f_{\tau(1)} \).
Suppose first that $f_{r(1)} \gtrsim 0$. Then

$$\begin{align*}
V[h_{\mathcal{A}c}f] &= w^-(\sum_{j=1}^{k} p_{o(j)})v(x_{s}) + \sum_{i=k+1}^{n} \left( w^-(\sum_{j=1}^{i} p_{o(j)}) - w^-(\sum_{j=1}^{i-1} p_{o(j)}) \right) v(h_{o(i)}) \\
& + \sum_{i=1}^{\vert \mathcal{A} \vert} \left( w^+(\sum_{j=i+1}^{\vert \mathcal{A} \vert} p_{r(j)}) - w^+(\sum_{j=i}^{\vert \mathcal{A} \vert} p_{r(j)}) \right) v(f_{r(i)})
\end{align*}$$

and

$$\begin{align*}
V[g_{\mathcal{A}c}f] &= w^-(\sum_{j=1}^{k} p_{o(j)})v(x_{s}) + \sum_{i=k+1}^{n} \left( w^-(\sum_{j=1}^{i} p_{o(j)}) - w^-(\sum_{j=1}^{i-1} p_{o(j)}) \right) v(g_{o(i)}) \\
& + \sum_{i=1}^{\vert \mathcal{A} \vert} \left( w^+(\sum_{j=i+1}^{\vert \mathcal{A} \vert} p_{r(j)}) - w^+(\sum_{j=i}^{\vert \mathcal{A} \vert} p_{r(j)}) \right) v(f_{r(i)})
\end{align*}$$

Since $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k + 2$, we have

$$g_{\mathcal{A}c}f \succ h_{\mathcal{A}c}f \iff V[g_{\mathcal{A}c}f] > V[h_{\mathcal{A}c}f] \iff \left( w^-(\sum_{j=1}^{k+1} p_{o(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)}) \right)(v(h_{o(k+1)}) - v(g_{o(k+1)}))$$

$$+ \left( w^-(\sum_{j=1}^{k+2} p_{o(j)}) - w^-(\sum_{j=1}^{k+1} p_{o(j)}) \right)(v(h_{o(k+2)}) - v(g_{o(k+2)})) < 0. \tag{8}$$

Conditional preferences are not represented by $E_L^m[-s \in \mathcal{A}^c]$ if we also have that $E_L^m[h\mid s \in \mathcal{A}^c] \geq E_L^m[g\mid s \in \mathcal{A}^c]$, which with $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k + 2$ reduces to

$$\left( w^-(\sum_{j=1}^{k+1} p_{o(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)}) \right)(v(h_{o(k+1)}) - v(g_{o(k+1)}))$$

$$+ \left( w^-(\sum_{j=1}^{k+2} p_{o(j)}) - w^-(\sum_{j=1}^{k+1} p_{o(j)}) \right)(v(h_{o(k+2)}) - v(g_{o(k+2)})) \geq 0. \tag{9}$$

Rearranging (8) and (9), conditional preferences are not represented by $E_L^m[-s \in \mathcal{A}^c]$ if

$$\frac{w^-(\sum_{j=1}^{k+2} p_{o(j)}) - w^-(\sum_{j=1}^{k+1} p_{o(j)})}{w^-(\sum_{j=1}^{k+1} p_{o(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)})} < \frac{v(g_{o(k+1)}) - v(h_{o(k+1)})}{v(h_{o(k+2)}) - v(g_{o(k+2)})}$$

$$\leq \frac{w^-(\sum_{j=1}^{k+2} p_{o(j)}) - w^-(\sum_{j=1}^{k+1} p_{o(j)})}{w^-(\sum_{j=1}^{k+1} p_{o(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)})}. \tag{10}$$

The functions $w^-(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (10) are satisfied.
Therefore, preferences are not represented as in statement (a) of Theorem 1.
Suppose second that $0 \succ f_{\tau(1)}$. Define

$$
\mu \equiv \begin{cases} 
\min \{ j \in \{1, \ldots, |\tilde{A}_L|\} : f_{\tau(j)} \succ f_{\tau(1)} \} & \text{if there exists } s \in \tilde{A}_L \text{ such that } f_s \succ f_{\tau(1)} \\
|\tilde{A}_L| + 1 & \text{if } f_s \sim f_{\tau(1)} \text{ for all } s \in \tilde{A}_L.
\end{cases}
$$

Since there are at least four outcomes in the loss domain, there exists $\hat{x}$ such that $f_{\tau(1)} \succ \hat{x} \succ x$, and/or there exists $x'$ such that $0 \succ x' \succ f_{\tau(1)}$. In the former case, since $0 \succ f_{\tau(1)}$, we can let $f_{\tau(\mu)} \succ h_{o(k+2)} \succ g_{o(k+2)} = g_{o(k+2)} \succ h_{o(k+1)} = x_*$. In the latter case, if it is also the case that $f_{\tau(\mu)} \succ \hat{x} \succ f_{\tau(1)}$ for some $\hat{x} \in X_L$, we can let $f_{\tau(\mu)} \succ h_{o(k+2)} \succ g_{o(k+2)} \succ f_{\tau(1)} = g_{o(k+2)} \succ h_{o(k+1)} = x_*$. In either of these two cases,

$$
V[h_{\mathcal{A} \mathcal{C} f}] = w^-(\sum_{j=1}^{k} p_{o(j)})v(x_*) + \left( w^-\left( \sum_{j=1}^{k+1} p_{o(j)} \right) - w^-\left( \sum_{j=1}^{k} p_{o(j)} \right) \right) v(h_{o(k+1)})
$$

$$
+ \sum_{i=1}^{\mu-1} \left( w^-\left( \sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)} \right) - w^-\left( \sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)} \right) \right) v(f_{\tau(i)})
$$

$$
+ \sum_{i=k+2}^{n} \left( w^-\left( \sum_{j=1}^{n} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)} \right) - w^-\left( \sum_{j=1}^{n} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)} \right) \right) v(h_{o(i)})
$$

$$
+ \sum_{i=\mu}^{\tilde{A}_L} \left( w^-\left( \sum_{j=1}^{n} p_{o(j)} + \sum_{j=1}^{i} p_{\tau(j)} \right) - w^-\left( \sum_{j=1}^{n} p_{o(j)} + \sum_{j=1}^{i-1} p_{\tau(j)} \right) \right) v(f_{\tau(i)})
$$

$$
+ \sum_{i=|\tilde{A}_L|+1}^{|\tilde{A}_L|} \left( w^+\left( \sum_{j=1}^{i} p_{\tau(j)} \right) - w^+\left( \sum_{j=i+1}^{|\tilde{A}_L|} p_{\tau(j)} \right) \right) v(f_{\tau(i)})
$$

(11)
and

\[
V[g_{A^c}f] = w^{-}\left(\sum_{j=1}^{k} p_{o(j)}v(x_*) + \left(w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)}\right) - w^{-}\left(\sum_{j=1}^{k} p_{o(j)}\right)\right)v(g_{o(k+1)})\right)
\]

\[
+ \sum_{i=1}^{\mu-1} \left(w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{r(j)}\right) - w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{r(j)}\right)\right)v(f_{r(i)})
\]

\[
+ \sum_{i=k+2}^{n} \left(w^{-}\left(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{r(j)}\right) - w^{-}\left(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{i-1} p_{r(j)}\right)\right)v(g_{o(i)})
\]

\[
+ \sum_{i=|\hat{A}|+1}^{n} \left(w^{+}\left(\sum_{j=1}^{i} p_{r(j)}\right) - w^{+}\left(\sum_{j=1}^{i} p_{r(j)}\right)\right)v(f_{r(i)}).
\]

(12)

Since \(h_{o(i)} = g_{o(i)} = h_{o(k+2)}\) for all \(i > k + 2\), we have

\[
g_{A^c}f \succ h_{A^c}f
\]

\[
\iff \quad V[g_{A^c}f] > V[h_{A^c}f]
\]

\[
\iff \quad \left(w^{-}\left(\sum_{j=1}^{k+2} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{r(j)}\right) - w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{r(j)}\right)\right)v(h_{o(k+2)}) - v(g_{o(k+2)})
\]

\[
+ \left(w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)}\right) - w^{-}\left(\sum_{j=1}^{k} p_{o(j)}\right)\right)v(h_{o(k+1)}) - v(g_{o(k+1)}) < 0.
\]

(13)

When \(g_{A^c}f \succ h_{A^c}f\), conditional preferences are not represented by \(E_{L}^{\text{opt}}[\cdot|s \in A^c]\) if (9) also holds. Hence, rearranging (9) and (13), conditional preferences are not represented by \(E_{L}^{\text{opt}}[\cdot|s \in A^c]\) if

\[
\frac{w^{-}\left(\sum_{j=1}^{k+2} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{r(j)}\right) - w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{\mu-1} p_{r(j)}\right)}{w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)}\right) - w^{-}\left(\sum_{j=1}^{k} p_{o(j)}\right)}
\]

\[
< \frac{v(h_{o(k+1)}) - v(g_{o(k+1)})}{v(h_{o(k+2)}) - v(g_{o(k+2)})} \leq \frac{w^{-}\left(\sum_{j=1}^{k+2} p_{o(j)}\right) - w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)}\right)}{w^{-}\left(\sum_{j=1}^{k+1} p_{o(j)}\right) - w^{-}\left(\sum_{j=1}^{k} p_{o(j)}\right)}.
\]

(14)

Again, the functions \(w^{-}\left(\cdot\right)\) and \(v(\cdot)\) can be chosen such that both inequalities in (14) are satisfied. Therefore, preferences are not represented as in statement (a) of Theorem 1.

If there does not exist \(\tilde{x} \in X_L\) such that \(f_{r(1)} \succ \tilde{x} \succ x_*\), and there does not exist \(\tilde{x} \in X_L\) such that \(f_{r(\mu)} \succ \tilde{x} \succ f_{r(1)}\), we can let \(h_{o(k+2)} \succ g_{o(k+2)} = f_{r(\mu)} \succ f_{r(1)} \succ g_{o(k+1)} \succ h_{o(k+1)} = x_*\).
which is possible since there are at least four outcomes in the loss domain. Define
\[
\xi \equiv \begin{cases} 
\min \left\{ j \in \{ 1, \ldots, |\hat{A}_L| \} : f_{\tau(j)} \succ f_{\tau(\mu)} \right\} & \text{if there exists } s \in \hat{A}_L \text{ such that } f_s \succ f_{\tau(\mu)} \\
|\hat{A}_L| + 1 & \text{if } f_{\tau(\mu)} \succeq f_s \text{ for all } s \in \hat{A}_L.
\end{cases}
\]
Then,
\[
V[h_{\hat{A}} f] = w^{-} \left( \sum_{j=1}^{k} p_{o(j)} v(x_s) + \left( w^{-} \left( \sum_{j=1}^{k+1} p_{o(j)} \right) - w^{-} \left( \sum_{j=1}^{k} p_{o(j)} \right) \right) v(h_{o(k+1)}) \right)
+ \sum_{i=1}^{\xi-1} \left( w^{-} \left( \sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{r(j)} \right) - w^{-} \left( \sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{r(j)} \right) \right) v(f_{\tau(i)})
+ \sum_{i=k+2}^{n} \left( w^{-} \left( \sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)} \right) - w^{-} \left( \sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)} \right) \right) v(h_{o(i)})
+ \sum_{i=|\hat{A}_L|+1}^{\hat{A}_L} \left( w^{+} \left( \sum_{j=1}^{\hat{A}_L} p_{r(j)} \right) - w^{+} \left( \sum_{j=|\hat{A}_L|+1}^{\hat{A}_L} p_{r(j)} \right) \right) v(f_{\tau(i)})
\]
and
\[
V[g_{\hat{A}} f] = w^{-} \left( \sum_{j=1}^{k} p_{o(j)} v(x_s) + \left( w^{-} \left( \sum_{j=1}^{k+1} p_{o(j)} \right) - w^{-} \left( \sum_{j=1}^{k} p_{o(j)} \right) \right) v(g_{o(k+1)}) \right)
+ \sum_{i=1}^{\xi-1} \left( w^{-} \left( \sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i} p_{r(j)} \right) - w^{-} \left( \sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{i-1} p_{r(j)} \right) \right) v(f_{\tau(i)})
+ \sum_{i=k+2}^{n} \left( w^{-} \left( \sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)} \right) - w^{-} \left( \sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)} \right) \right) v(g_{o(i)})
+ \sum_{i=|\hat{A}_L|}^{\hat{A}_L} \left( w^{+} \left( \sum_{j=1}^{\hat{A}_L} p_{r(j)} \right) - w^{+} \left( \sum_{j=|\hat{A}_L|}^{\hat{A}_L} p_{r(j)} \right) \right) v(f_{\tau(i)})
+ \sum_{i=|\hat{A}_L|+1}^{\hat{A}_L} \left( w^{+} \left( \sum_{j=1}^{\hat{A}_L} p_{r(j)} \right) - w^{+} \left( \sum_{j=|\hat{A}_L|+1}^{\hat{A}_L} p_{r(j)} \right) \right) v(f_{\tau(i)}).
Since $h_{o(i)} = g_{o(i)} = h_{o(k+2)}$ for all $i > k + 2$, we have

$$V[g_A \circ f] > V[h_A \circ f]$$

$$\iff \left( w^-(\sum_{j=1}^{k+2} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)}) - w^-\left(\sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)}\right)\right) (v(h_{o(k+2)}) - v(g_{o(k+2)}))$$

$$+ \left( w^-(\sum_{j=1}^{k+1} p_{o(j)}) - w^-\left(\sum_{j=1}^{k} p_{o(j)}\right)\right) (v(h_{o(k+1)}) - v(g_{o(k+1)})) < 0.$$  \hfill (15)

Combining (15) with (9), conditional preferences are not represented by $E_L^m[\cdot | s \in A^C]$ if

$$w^-(\sum_{j=1}^{k+1} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)}) - w^-\left(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\xi-1} p_{r(j)}\right)$$

$$\leq \frac{v(g_{o(k+1)}) - v(h_{o(k+1)})}{v(h_{o(k+2)}) - v(g_{o(k+2)})} \leq \frac{w^-(\sum_{j=1}^{k+2} p_{o(j)}) - w^-\left(\sum_{j=1}^{k+1} p_{o(j)}\right)}{w^-\left(\sum_{j=1}^{k+1} p_{o(j)}\right) - w^-\left(\sum_{j=1}^{\xi-1} p_{r(j)}\right)}.$$  \hfill (16)

The functions $w^-(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (16) are satisfied. Therefore, preferences are not represented as in statement (a) of Theorem 1. The cases considered exhaust all possibilities for $f_s \succ x$ for some $s \in \hat{A}$.

The second part of the necessity proof is to show that if $0 \succ f_s$ for some $s \in A^C$, then we can have $g_A f \succ h_A f$ but $E_L^c[h|s \in A] \geq E_L^c[g|s \in A]$. Since there are at least four outcomes in the loss domain, we can choose $h, g \in \mathcal{H}_L$ such that

$$0 \succ h_{o(k)} \succ g_{o(k)} \succ g_{o(k-1)} \succ h_{o(k-1)} \succ x^*$$

and $h_{o(i)} = g_{o(i)} = h_{o(k-1)}$ for all $i < k - 1$. Again, depending on how $f$ relates to the outcomes in the loss domain, there are different cases we need to consider.

The procedure is essentially the same as above: In each case, we calculate the utility of $h_A f$ and $g_A f$ and derive the conditions for $g_A f \succ h_A f$ and $E_L^c[h|s \in A] \geq E_L^c[g|s \in A]$. We then show existence of a $w^-(\cdot)$ and $v(\cdot)$ such that both conditions are satisfied.

Let $\hat{A} = \{s \in A^C | 0 \succ f_s\}$ and $|\hat{A}|$ denote the number of states in $\hat{A}$. Let $\beta$ be a function defined on $\{1, \ldots, |\hat{A}|\}$ that orders the states in $\hat{A}$ such that $f_{\beta(|\hat{A}|)} \succ \cdots \succ f_{\beta(1)}$. Define

$$\eta \equiv \left\{ \begin{array}{ll} \min \{ j \in \{1, \ldots, |\hat{A}|\} : f_{\beta(j)} \succ f_{\beta(1)} \} & \text{if there exists } s \in \hat{A} \text{ such that } f_s \succ f_{\beta(1)} \\ |\hat{A}| + 1 & \text{if } f_s \sim f_{\beta(1)} \text{ for all } s \in \hat{A}, \end{array} \right.$$  

$$\rho \equiv \left\{ \begin{array}{ll} \min \{ j \in \{1, \ldots, |\hat{A}|\} : f_{\beta(j)} \succ f_{\beta(\eta)} \} & \text{if there exists } s \in \hat{A} \text{ such that } f_s \succ f_{\beta(\eta)} \\ |\hat{A}| + 1 & \text{if } f_{\beta(\eta)} \succ f_{s} \text{ for all } s \in \hat{A}, \end{array} \right.$$  

and

$$\sigma \equiv \left\{ \begin{array}{ll} \min \{ j \in \{1, \ldots, |\hat{A}|\} : f_{\beta(j)} \succ f_{\beta(\rho)} \} & \text{if there exists } s \in \hat{A} \text{ such that } f_s \succ f_{\beta(\rho)} \\ |\hat{A}| + 1 & \text{if } f_{\beta(\rho)} \succ f_{s} \text{ for all } s \in \hat{A}. \end{array} \right.$$  

Suppose first that $f_{\beta(1)} = x^*$. If there exist $\hat{z}, \hat{z} \in X_L$ such that $f_{\beta(\eta)} \succ \hat{z} \succ \hat{z} \succ x^*$, we can
let \( f_\beta(\eta) \gtrsim h_o(k) \gtrsim g_o(k) \gtrsim h_o(k-1) = x_* \). Then,

\[
V[h_Af] = w^-(\sum_{j=1}^{\eta-1} p_\beta(j))v(f_\beta(1)) + \sum_{i=1}^{\eta-1} \left( \sum_{j=1}^{\eta-1} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j) \right) - w^-(\sum_{j=1}^{\eta-1} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j))v(h_o(i)) \\
+ \sum_{i=\eta+1}^{\eta+\eta} \left( \sum_{j=1}^{\eta+1} p_o(j) + \sum_{j=1}^{\eta+1} p_\beta(j) \right) - w^-(\sum_{j=1}^{\eta+1} p_o(j) + \sum_{j=1}^{\eta+1} p_\beta(j))v(f_\beta(i)) \\
+ \sum_{i=k+1}^{k+\eta} \left( \sum_{j=i}^{\eta} p_o(j) - w^+(\sum_{j=i}^{\eta} p_\beta(j)) \right)v(f_\beta(i))
\]

and

\[
V[g_Af] = w^-(\sum_{j=1}^{\eta-1} p_\beta(j))v(f_\beta(1)) + \sum_{i=1}^{\eta-1} \left( \sum_{j=1}^{\eta-1} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j) \right) - w^-(\sum_{j=1}^{\eta-1} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j))v(g_o(i)) \\
+ \sum_{i=\eta+1}^{\eta+\eta} \left( \sum_{j=1}^{\eta+1} p_o(j) + \sum_{j=1}^{\eta+1} p_\beta(j) \right) - w^-(\sum_{j=1}^{\eta+1} p_o(j) + \sum_{j=1}^{\eta+1} p_\beta(j))v(f_\beta(i)) \\
+ \sum_{i=k+1}^{k+\eta} \left( \sum_{j=i}^{\eta} p_o(j) - w^+(\sum_{j=i}^{\eta} p_\beta(j)) \right)v(f_\beta(i)).
\]

Thus, since \( h_o(i) = g_o(i) = h_o(k-1) \) for all \( i < k-1 \), we have

\[
V[g_Af] > V[h_Af] \\
\iff \left( w^-(\sum_{j=1}^{\eta-1} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j)) - w^-(\sum_{j=1}^{\eta-1} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j)) \right) (v(h_o(k-1)) - v(g_o(k-1))) \\
+ \left( w^-(\sum_{j=1}^{k} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j)) - w^-(\sum_{j=1}^{k} p_o(j) + \sum_{j=1}^{\eta-1} p_\beta(j)) \right) (v(h_o(k)) - v(g_o(k))) < 0.
\]

These preferences are not represented by \( E^x_1|s \in A \) if \( E^x_1[h|s \in A] \geq E^x_1[g|s \in A] \), which with \( h_o(i) = g_o(i) = h_o(k-1) \) for all \( i < k-1 \) reduces to

\[
\left( w^-(\sum_{j=1}^{k-1} p_o(j) - w^-(\sum_{j=1}^{k-1} p_o(j)) \right) (v(h_o(k-1)) - v(g_o(k-1))) \\
+ \left( w^-(\sum_{j=1}^{k} p_o(j) - w^-(\sum_{j=1}^{k-1} p_o(j)) \right) (v(h_o(k)) - v(g_o(k))) \right) \geq 0. \quad (17)
\]
We therefore have that conditional preferences are not represented by $E_s^*\{\cdot|s \in A\}$ if
\[
\frac{w^{-}(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)})}{w^{-}(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)})} < \frac{v(g_{o(k-1)}) - v(h_{o(k-1)})}{v(h_{o(k)}) - v(g_{o(k)})} \leq \frac{w^{-}(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)})}{w^{-}(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{k-2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)})},
\]
(18)

As above, the functions $w^{-}(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (18) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

If there exist $\bar{z}, \tilde{z} \in X_L$ such that $\bar{z} \succ f_{\beta(\eta)} \succ \tilde{z} \succ x_s$, we can let $h_{o(k)} = \bar{z}$, $g_{o(k)} = f_{\beta(\eta)}$, $g_{o(k-1)} = \tilde{z}$, and $h_{o(k-1)} = x_s$. If there exist $\bar{z}, \tilde{z} \in X_L$ such that $f_{\beta(\rho)} \succ \bar{z} \succ \tilde{z} \succ f_{\beta(\eta)}$, we can let $h_{o(k)} = \tilde{z}$, $g_{o(k)} = \bar{z}$, $g_{o(k-1)} = f_{\beta(\eta)}$, and $h_{o(k-1)} = x_s$. In either of these cases,
\[
V[h_{Af}] = \frac{w^{-}(\sum_{j=1}^{k} p_{\beta(j)}) v(f_{\beta(1)}) + \sum_{i=1}^{k-1} \left( w^{-}(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) v(h_{o(i)})}{\sum_{i=1}^{k-1} \left( w^{-}(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) v(f_{\beta(i)})} + \sum_{i=k+1}^{n} \left( w^{+}(\sum_{j=i}^{n} p_{\beta(j)}) - w^{+}(\sum_{j=i+1}^{n} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]
(19)
and
\[
V[g_{Af}] = \frac{w^{-}(\sum_{j=1}^{\eta-1} p_{\beta(j)}) v(f_{\beta(1)}) + \sum_{i=1}^{k-1} \left( w^{-}(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) v(g_{o(i)})}{\sum_{i=1}^{k-1} \left( w^{-}(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^{-}(\sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) v(f_{\beta(i)})} + \sum_{i=k+1}^{n} \left( w^{+}(\sum_{j=i}^{n} p_{\beta(j)}) - w^{+}(\sum_{j=i+1}^{n} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]
(20)
Therefore, since $h_{o(i)} = g_{o(i)} = h_{o(k-1)}$ for all $i < k - 1$, we have

$$V[g_Af] > V[h_Af]$$

$$\Leftrightarrow \left( w^{-} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k-2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) \right) \left( v(h_{o(k-1)}) - v(g_{o(k-1)}) \right)$$

$$+ \left( w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) \right) \left( v(h_{o(k)}) - v(g_{o(k)}) \right) < 0.$$  

Together with (17) we have that conditional preferences are not represented by $E_L^s[\cdot|s \in A]$ if

$$\frac{w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right)}{v(h_{o(k)}) - v(g_{o(k)})} < \frac{w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k-1} p_{o(j)} \right)}{v(h_{o(k-1)}) - v(g_{o(k-1)})}. \tag{21}$$

Again, the functions $w^{-}(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (21) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

Finally, if there does not exist $\tilde{z} \in X_L$ such that $f_{\beta(\eta)} > \tilde{z} > x_*$ and there does not exist $\hat{z} \in X_L$ such that $f_{\beta(\rho)} > \hat{z} > f_{\beta(\eta)}$, then, since there are at least four outcomes in the loss domain, there exists $\tilde{z} \in X_L$ such that $\tilde{z} > f_{\beta(\rho)}$. In this case, we can let $g_{o(k-1)} = f_{\beta(\eta)}$ and $g_{o(k)} = f_{\beta(\rho)}$. Then,

$$V[h_Af] = w^{-} \left( \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) v(f_{\beta(1)}) + \sum_{i=1}^{k-1} \left( w^{-} \left( \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) \right) v(h_{o(i)})$$

$$+ \left( w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) \right) v(f_{\beta(\eta)})$$

$$+ \left( w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) \right) v(f_{\beta(\rho)})$$

$$+ \left( w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) \right) v(h_{o(k)})$$

$$+ \sum_{i=\sigma}^{\hat{A}} \left( w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) - w^{-} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)} \right) \right) v(f_{\beta(i)})$$

$$+ \sum_{i=k+1}^{n} \left( w^{+} \left( \sum_{j=1}^{n} p_{\beta(j)} \right) - w^{+} \left( \sum_{j=1}^{k} p_{\beta(j)} \right) \right) v(f_{\beta(i)})$$
and

\[
V[g_Af] = w^-(\sum_{j=1}^{\eta-1} p_{\beta(j)})v(f_{\beta(1)}) + \sum_{i=1}^{k-1} \left( w^-(\sum_{j=1}^{i} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right)v(g_{o(i)})
+ \left( w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right)v(f_{\beta(\eta)})
+ \left( w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) \right)v(f_{\beta(\rho)})
+ \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) \right)v(g_{o(k)})
+ \sum_{i=\sigma}^{|\hat{A}|} \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}) \right)v(f_{\beta(i)})
+ \sum_{i=k+1}^{n} \left( w^+(\sum_{j=1}^{n} p_{\beta(j)}) - w^+(\sum_{j=1}^{k} p_{\beta(j)}) \right)v(f_{\beta(i)}).
\]

Thus, since \( h_{o(i)} = g_{o(i)} = h_{o(k-1)} \) for all \( i < k - 1 \), we have

\[
g_Af \succ h_Af
\]

\[
\Leftrightarrow V[g_Af] > V[h_Af]
\]

\[
\Leftrightarrow \left( w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right)\left( v(h_{o(k-1)}) - v(g_{o(k-1)}) \right)
+ \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) \right)\left( v(h_{o(k)}) - v(g_{o(k)}) \right) < 0.
\]

This and (17) imply that conditional preferences are not represented by \( E^s_L[\cdot|s \in A] \) if

\[
\frac{w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)})}{w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-2} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)})} < \frac{v(g_{o(k-1)}) - v(h_{o(k-1)})}{v(h_{o(k)}) - v(g_{o(k)})} \leq \frac{w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\sigma-1} p_{\beta(j)})}{w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-2} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)})}.
\]

(22)

The functions \( w^-(\cdot) \) and \( v(\cdot) \) can, once again, be chosen such that both inequalities in (22) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

Suppose second that there exist \( \hat{x} \) such that \( f_{\beta(1)} \succ \hat{x} \succ x_{*} \). Then we can let \( f_{\beta(\eta)} \succ h_{o(k)} \succ
\[ g_0(k) = f_{\beta(1)} \succ g_0(k-1) \succ h_0(k-1). \] We thus have that

\[
V[h_Af] = w^-(\sum_{j=1}^{k-2} p_{o(j)})v(h_{o(k-1)}) + \left( w^-(\sum_{j=1}^{k-1} p_{o(j)}) - w^-(\sum_{j=1}^{k-2} p_{o(j)}) \right) v(h_{o(k-1)})
\]

\[
+ \left( w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)}) \right) v(f_{\beta(1)})
\]

\[
+ \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) v(h_{o(k)})
\]

\[
+ \sum_{i=\eta}^{\vert \hat{A} \vert} \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]

\[
+ \sum_{i=k+1}^{n} \left( w^+(\sum_{j=1}^{n} p_{\beta(j)} - w^+(\sum_{j=1}^{k} p_{\beta(j)}) \right) v(f_{\beta(i)}).
\] (23)

and

\[
V[g_Af] = w^-(\sum_{j=1}^{k-2} p_{o(j)})v(h_{o(k-1)}) + \left( w^-(\sum_{j=1}^{k-1} p_{o(j)}) - w^-(\sum_{j=1}^{k-2} p_{o(j)}) \right) v(g_{o(k+1)})
\]

\[
+ \left( w^-(\sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k-1} p_{o(j)}) \right) v(f_{\beta(1)})
\]

\[
+ \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) v(g_{o(k)})
\]

\[
+ \sum_{i=\eta}^{\vert \hat{A} \vert} \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]

\[
+ \sum_{i=k+1}^{n} \left( w^+(\sum_{j=1}^{n} p_{\beta(j)} - w^+(\sum_{j=1}^{k} p_{\beta(j)}) \right) v(f_{\beta(i)}).
\] (24)

Hence,

\[ g_Af \succ h_Af \]

\[ \Leftrightarrow V[g_Af] > V[h_Af] \]

\[ \Leftrightarrow \left( w^-(\sum_{j=1}^{k-1} p_{o(j)}) - w^-(\sum_{j=1}^{k-2} p_{o(j)}) \right) \left( v(h_{o(k-1)}) - v(g_{o(k-1)}) \right) \]

\[ + \left( w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-(\sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) \right) \left( v(h_{o(k)}) - v(g_{o(k)}) \right) < 0. \] (25)

Conditional preferences are not represented by \( E'_L[\vert s \in A] \) if (17) holds. Combining (25) and
(17), we have that preferences are not represented by $E_L^P[|s \in A]$ if
\[
\frac{w^-( \sum_{j=1}^k p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-( \sum_{j=1}^{\rho-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)})}{w^-( \sum_{j=1}^{\eta-1} p_{o(j)}) - w^-( \sum_{j=1}^{\rho-1} p_{o(j)})} < \frac{v(g_{o(k-1)}) - v(h_{o(k-1)})}{v(h_{o(k)}) - v(g_{o(k)})} \leq \frac{w^-( \sum_{j=1}^k p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)}) - w^-( \sum_{j=1}^{\rho-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)})}{w^-( \sum_{j=1}^{\eta-1} p_{o(j)}) - w^-( \sum_{j=1}^{\rho-1} p_{o(j)})}.
\]
(26)

As above, the functions $w^-(\cdot)$ and $v(\cdot)$ can be chosen such that both inequalities in (26) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem.

Suppose finally that $f_{\beta(1)} > x_*$ and there does not exist $\hat{x}$ such that $f_{\beta(1)} > \hat{x} > x_*$. Then, since there are at least four outcomes in the gain domain, there exists $x''$ such that $0 > x'' > f_{\beta(1)}$. Therefore, we can choose $h$ and $g$ such that $h_{o(k)} \succ g_{o(k)} \succ f_{\beta(1)}$ and $g_{o(k-1)} = f_{\beta(1)} \succ h_{o(k-1)}$. If there exists $\tilde{x}$ in the loss domain such that $f_{\beta(1)} \succ \tilde{x} \succ f_{\beta(1)}$, we can let $f_{\beta(1)} \succ h_{o(k)}$. Then the utility of $h_A f$ is given by (23) and the utility of $g_A f$ is given by (24). Therefore, preferences are not represented by $E_L^P[|s \in A]$ if the inequalities in (26) hold, and again it is possible to choose the functions $w^-(\cdot)$ and $v(\cdot)$ such that both inequalities are satisfied.

If there does not exist $\tilde{x}$ in the loss domain such that $f_{\beta(1)} \succ \tilde{x} \succ f_{\beta(1)}$, we can, again because the loss domain consists of at least four outcomes, let $g_{o(k)} = h_{o(k)} = f_{\beta(1)}$. Define $\rho = \min\{j | f_{\beta(j)} \succ f_{\beta(1)}\}$. Then
\[
V[h_A f] = w^-( \sum_{j=1}^{k-2} p_{o(j)} ) v(h_{o(k-1)}) + \left( w^-( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}) - w^-( \sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}) \right) v(h_{o(k-1)})
\]
\[
+ \left( w^-( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}) - w^-( \sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{i-1} p_{\beta(j)}) \right) v(h_{o(k)})
\]
\[
+ \sum_{i=\rho}^{\rho-1} \left( w^-( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) - w^-( \sum_{j=1}^{\rho-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]
\[
+ \sum_{i=\rho}^{n} \left( w^+( \sum_{j=1}^{i-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) - w^+( \sum_{j=1}^{\rho-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]
\[
+ \sum_{i=k+1}^{n} \left( w^+( \sum_{j=i+1}^{n} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) - w^+( \sum_{j=1}^{\rho-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)}) \right) v(f_{\beta(i)})
\]
\[\cdot|\tilde{i} \notin A\].
and
\[
V[g_A f] = w^{-k-2} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) v(h_{o(k-1)}) + \left( w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) - w^{-k-2} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) \right) v(g_{o(k-1)}) \\
+ \left( w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) - w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\eta-1} p_{\beta(j)} \right) \right) v(f_{\beta(1)}) \\
+ \left( w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) \right) v(f_{\beta(\eta)}) \\
+ \left( w^{-k} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) - w^{-k} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) \right) v(f_{\beta(i)}) \\
+ \sum_{i=\rho}^{n} \left( w^{n} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) - w^{n} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) \right) v(f_{\beta(i)}).
\]

Therefore,
\[
g_A f \succ h_A f \\
\Leftrightarrow V[g_A f] > V[h_A f] \\
\Leftrightarrow \left( w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) - w^{-k-2} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) \right) \left( v(h_{o(k-1)}) - v(g_{o(k-1)}) \right) \\
+ \left( w^{-k-1} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) - w^{-k-1} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{i} p_{\beta(j)} \right) \right) \left( v(h_{o(k)}) - v(g_{o(k)}) \right) < 0.
\]

Combining this with (17) we have that preferences are not represented by \( E_L^v : |s \in A \) if
\[
\frac{w^{-k} \left( \sum_{j=1}^{k} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right) - w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} + \sum_{j=1}^{\rho-1} p_{\beta(j)} \right)}{w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) - w^{-k-2} \left( \sum_{j=1}^{k-2} p_{o(j)} \right)} \\
< \frac{v(g_{o(k-1)}) - v(h_{o(k-1)})}{v(h_{o(k)}) - v(g_{o(k)})} \leq \frac{w^{-k} \left( \sum_{j=1}^{k} p_{o(j)} \right) - w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} \right)}{w^{-k-1} \left( \sum_{j=1}^{k-1} p_{o(j)} \right) - w^{-k-2} \left( \sum_{j=1}^{k-2} p_{o(j)} \right)}. \tag{27}
\]

Again, the functions \( w^{-} \) and \( v(\cdot) \) can be chosen such that both inequalities in (27) are satisfied. Therefore, preferences are not represented as in statement (a) of the theorem. ■