Lecture 6: Options
Key concepts

_ Introduction to options
_ Option payoffs
_ Corporate securities as options
_ Use of options
_ Basic properties of options
_ Binomial Option Pricing Model
_ Black-Scholes option pricing formula

Readings:
_ Brealey, Myers and Allen, Chapters 21 - 22
_ Bodie, Kane and Markus, Chapters 20 - 21
Option types:

**Call:** The right to buy an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date)

**Put:** The right to sell an asset for a given price on or before the expiration date

Exercise styles:

European: Owner can exercise the option only on expiration date

American: Owner can exercise the option on or before expiration date

Key elements in defining an option:

- Underlying asset and its price $S$
- Exercise price (strike price) $K$
- Expiration date (maturity date) $T$ (today is 0)
- European or American
Example. A European call option on IBM with exercise price $100. It gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at $100 on the expiration date. Depending on the share price of IBM on the expiration date, the option's payoff is:

<table>
<thead>
<tr>
<th>IBM Price at T</th>
<th>Action</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Not Exercise</td>
<td>0</td>
</tr>
<tr>
<td>80</td>
<td>Not Exercise</td>
<td>0</td>
</tr>
<tr>
<td>90</td>
<td>Not Exercise</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>Not Exercise</td>
<td>0</td>
</tr>
<tr>
<td>110</td>
<td>Exercise</td>
<td>10</td>
</tr>
<tr>
<td>120</td>
<td>Exercise</td>
<td>20</td>
</tr>
<tr>
<td>130</td>
<td>Exercise</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>Exercise</td>
<td>$T - 100$</td>
</tr>
</tbody>
</table>

- The payoff of an option is never negative. Sometimes, it is positive.
- Actual payoff depends on the price of the underlying asset:

\[
CF_T \text{ (call)} = \max [S_T - K, 0]
\]
Option payoffs can be plotted as a function of the price of the underlying asset at expiration:

- **Payoff of buying a call**
  - If the asset price is below the strike price, the payoff is 0.
  - If the asset price is above the strike price, the payoff is the asset price minus the strike price.

- **Payoff of buying a put**
  - If the asset price is above the strike price, the payoff is 0.
  - If the asset price is below the strike price, the payoff is the strike price minus the asset price.

- **Payoff of selling a call**
  - The payoff is the difference between the strike price and the asset price.

- **Payoff of selling a put**
  - The payoff is the difference between the asset price and the strike price.
The net payoff from an option must includes its cost.

**Example.** A European call on IBM shares with an exercise price of $100 and maturity of three months is trading at $5. The 3-month interest rate, not annualized, is 0.5%. What is the price of IBM that makes the call break-even?

At maturity, the call's net payoff is as follows:

<table>
<thead>
<tr>
<th>IBM Price</th>
<th>Action</th>
<th>Payoff</th>
<th>Net payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>:</td>
<td>Not Exercise</td>
<td>0</td>
<td>- 5.025</td>
</tr>
<tr>
<td>80</td>
<td>Not Exercise</td>
<td>0</td>
<td>- 5.025</td>
</tr>
<tr>
<td>90</td>
<td>Not Exercise</td>
<td>0</td>
<td>- 5.025</td>
</tr>
<tr>
<td>100</td>
<td>Not Exercise</td>
<td>0</td>
<td>- 5.025</td>
</tr>
<tr>
<td>110</td>
<td>Exercise</td>
<td>10</td>
<td>4.975</td>
</tr>
<tr>
<td>120</td>
<td>Exercise</td>
<td>20</td>
<td>14.975</td>
</tr>
<tr>
<td>130</td>
<td>Exercise</td>
<td>30</td>
<td>24.975</td>
</tr>
<tr>
<td>:</td>
<td>Exercise</td>
<td>$S_T - 100$</td>
<td>$S_T - 100 - 5.25$</td>
</tr>
</tbody>
</table>
The break even point is given by:

\[
\text{Net payoff} = \max[S_T - K, 0] - C(1 + r)^T \\
= S_T - 100 - (5)(1 + 0.005) \\
= 0
\]

or

\[
S_T = 105.025
\]
### Call option (price = C)

<table>
<thead>
<tr>
<th></th>
<th>if $S &lt; K$</th>
<th>if $S = K$</th>
<th>if $S &gt; K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>0</td>
<td>0</td>
<td>$S - K$</td>
</tr>
<tr>
<td>Profit</td>
<td>$-C(1+r)^T$</td>
<td>$-C(1+r)^T$</td>
<td>$S - K - C(1+r)^T$</td>
</tr>
</tbody>
</table>

### Put option (price = P)

<table>
<thead>
<tr>
<th></th>
<th>if $S &lt; K$</th>
<th>if $S = K$</th>
<th>if $S &gt; K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>$K - S$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Profit</td>
<td>$K - S - P(1+r)^T$</td>
<td>$-P(1+r)^T$</td>
<td>$-P(1+r)^T$</td>
</tr>
</tbody>
</table>
Using the payoff diagrams, we can also examine the payoff of a portfolio consisting of options as well as other assets.

**Example.** The underlying asset and the bond (with face value $100) have the following payoff diagram:
Stock + put

- **Buy stock**: The payoff increases linearly with the stock price.
- **Buy a put**: The payoff decreases linearly with the stock price.
- **Stock + put**: The payoff is a combination of the two strategies, showing a flat line at the strike price and then increasing linearly above it.
Call 1 - Call 2

- **Buy call with**
  - $K = 50$

- **Write call with**
  - $K = 60$

**Call$_1$ – Call$_2$**

- **Payoff**
  - **Stock price**

- Payoff graph showing the payoff for each strategy based on the stock price.
Call + Put

![Diagram showing payoff for buying a call and a put with strike price K = 50. The call pays off when the stock price is above the strike price, and the put pays off when the stock price is below the strike price. The combined payoff of holding both the call and the put shows a V-shaped curve, indicating profit when the stock price is away from the strike price.](image)
Example. Consider two firms, A and B, with identical assets but different capital structures (in market value terms).

<table>
<thead>
<tr>
<th>Balance sheet of A</th>
<th>Balance sheet of B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset</td>
<td>Asset</td>
</tr>
<tr>
<td>$30</td>
<td>$30</td>
</tr>
<tr>
<td>Bond</td>
<td>$25 Bond</td>
</tr>
<tr>
<td>0</td>
<td>$25</td>
</tr>
<tr>
<td>Equity</td>
<td>Equity</td>
</tr>
<tr>
<td>30 Equity</td>
<td>5 Equity</td>
</tr>
<tr>
<td>$30</td>
<td>$30</td>
</tr>
</tbody>
</table>

Firm B's bond has a face value of $50. Thus default is likely.
Example. (Cont’d)

Consider the value of stock A, stock B, and a call on the underlying asset of firm B with an exercise price of $50:

<table>
<thead>
<tr>
<th>Asset value</th>
<th>Value of stock A</th>
<th>Value of stock B</th>
<th>Value of call</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>: :</td>
<td>: :</td>
<td>: :</td>
</tr>
<tr>
<td>$20</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>80</td>
<td>80</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>: :</td>
<td>: :</td>
<td>: :</td>
</tr>
</tbody>
</table>

Stock B gives the same payoff as a call option written on its asset

Thus B's common stocks really are call options of firm's asset
Indeed, many corporate securities can be viewed as options:

**Common stock:** A call option on the assets of the firm with the exercise price being its bond's redemption value.

**Bond:** A portfolio combining the firm's assets and a short position in the call with exercise price equal bond redemption value.

\[
\text{Equity} \equiv \max [0, A - B] \\
\text{Debt} \equiv \min [A, B] = A - \max [0, A - B] \\
A = D + E
\]
Warrant: Call options on the stock issued by the firm.

Convertible bond: A portfolio combining straight bonds and a call option on the firm's stock with the exercise price related to the conversion ratio.

Callable bond: A portfolio combining straight bonds and a call written on the bonds.
For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.

Notation:

- $S$: Price of stock now
- $S_T$: Price of stock at $T$
- $B$: Price of discount bond of par $\$1$ and maturity $T$ ($B \leq 1$)
- $C$: Price of a European call with strike $K$ and maturity $T$
- $P$: Price of a European put with strike $K$ and maturity $T$

For our discussions:
- Consider only European options (no early exercise)
- Assume no dividends (option cash flow occurs only at maturity)
First consider European options on a non-dividend paying stock.

1. \( C \geq 0 \)
2. \( C \leq S \)

The payoff of stock dominates that of call:
3. $C \geq S - KB$

   Strategy (a): Buy a call
   Strategy (b): Buy a share of stock by borrowing KB

   The payoff of strategy (a) dominates that of strategy (b):

Since $C \geq 0$, we have $C \geq \max[S - KB, 0]$
4. Combining the above, we have

$$\max [S - KB, 0] \leq C \leq S$$
Put-call parity

Portfolio 1: A call with strike $100 and a bond with par $100
Portfolio 2: A put with strike $100 and a share of the underlying asset

Their payoffs are identical, so must be their prices:

\[ C + \frac{K}{(1+r)^T} = P + S \]

This is called the put-call parity.
Option value and volatility

Option value increases with the volatility of underlying asset.

**Example.** Two firms, A and B, with the same current price of $100. B has higher volatility of future prices. Consider call options written on A and B, respectively, with the same exercise price $100.

<table>
<thead>
<tr>
<th>Probability</th>
<th>Good state</th>
<th>bad state</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock A</td>
<td>120</td>
<td>80</td>
</tr>
<tr>
<td>Stock B</td>
<td>150</td>
<td>50</td>
</tr>
<tr>
<td>Call on A</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>Call on B</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

Clearly, call on stock B should be more valuable.
Determinants of option value:

Key factors in determining option value:
1. price of underlying asset $S$
2. strike price $K$
3. time to maturity $T$
4. interest rate $r$
5. volatility of underlying asset $\sigma$

Additional factors that can sometimes influence option value:
6. expected return on the underlying asset
7. investors' attitude toward risk, ...
In order to have a complete option pricing model, we need to make additional assumptions about:

1. Price process of the underlying asset (stock)
2. Other factors

We will assume, in particular, that:

- Prices do not allow arbitrage
- Prices are "reasonable"
- A benchmark model --- Price follows a binomial process.

\[ S_0 \quad \begin{cases} S_{up} \\ S_{down} \end{cases} \]

At \( t=0 \) and \( t=1 \), time
**Example.** Valuation of a European call on a stock.

- Current stock price is $50
- There is one period to go
- Stock price will either go up to $75 or go down to $25
- There are no cash dividends
- The strike price is $50
- One period borrowing and lending rate is 10%

The stock and bond present two investment opportunities:

\[
\begin{array}{c}
50 \quad 75 \\
\hline \\
25
\end{array}
\quad
\begin{array}{c}
1 \quad 1.1 \\
\hline \\
1.1
\end{array}
\]

The option's payoff at expiration is:

\[
C_0 \quad 25 \\
\hline \\
0
\]

What is \(C_0\), the value of the option today?
Form a portfolio of stock and bond that replicates the call’s payoff:
  ➢ a shares of the stock
  ➢ b dollars in the riskless bond
such that

\[
75a + 1.1b = 25 \\
25a + 1.1b = 0
\]

Unique solution: \( a = 0.5 \) and \( b = -11.36 \)

That is
  – buy half a share of stock and sell $11.36 worth of bond
  – payoff of this portfolio is identical to that of the call
  – present value of the call must equal the current cost of this `replicating portfolio" which is

\[
(50)(0.5) - 11.36 = 13.64
\]

Number of shares needed to replicate one call option is called the option’s hedge ratio or delta.

In the above problem, the option’s delta is \( a = 1/2 \).
More than one period:

Call price process:

- terminal value of the call is known, and
- \( C_u \) and \( C_d \) denote the option value next period when the stock price goes up and goes down, respectively
- Compute the current value by working backwards: first \( C_u \) and \( C_d \) and then \( C \)
Step 1. Start with Period 1:

1. Suppose the stock price goes up to $75 in period 1:
   _ Construct the replicating portfolio at node (t =1, up):
     \[ 112.5a + 1.1b = 62.5 \]
     \[ 37.5a + 1.1b = 0 \]
   _ Unique solution: a = 0.833, b = -28.4
   _ The cost of this portfolio: \((0.833)(75) - 28.4 = 34.075\)
   _ The exercise value of the option: \(75 - 50 = 25 \leq 34.075\)
   _ Thus, \(C_u = 34.075\).

2. Suppose the stock price goes down to $25 in period 1. Repeat the above for node (t =1, down):
   _ The replicating portfolio: a = 0, b = 0
   _ The call value at the lower node next period is \(C_d = 0\).
Step 2. Now go back one period, to Period 0:

- The option's value next period is either 34.075 or 0:

\[
C_0 \begin{cases} 
C_u = 34.075 \\
C_d = 0 
\end{cases}
\]

- If we can construct a portfolio of the stock and bond to replicate the value of the option next period, then the cost of this replicating portfolio must equal the option's present value.

- Find a and b so that

\[
\begin{align*}
75a + 1.1b &= 34.075 \\
25a + 1.1b &= 0
\end{align*}
\]

- Unique solution: a = 0.6815, b = -15.48

- The cost of this portfolio: (0.6815)(50) – 15.48 = 18.59

- The present value of the option must be \( C_0 = 18.59 \)

- It is greater than the exercise value 0 (thus no early exercise)
Play Forward:

Period 0: Spend $18.59 and borrow $15.48 at 10% interest rate to buy 0.6815 shares of the stock

Period 1:
- When the stock price goes up, the portfolio value becomes 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%
  - One period later, the payoff of this portfolio exactly matches that of the call
- When the stock price goes down, the portfolio becomes worthless. Close out the position.
  - The portfolio payoff one period later is zero
Thus

- No early exercise.
- Replicating strategy gives payoffs identical to those of the call.
- Initial cost of the replicating strategy must equal the call price.
What we have **used** to calculate option's value:
- current stock price
- magnitude of possible future changes of stock price -- volatility
- interest rate
- strike price
- time to maturity

What we have **not used**:
- probabilities of upward and downward movements
- investor's attitude towards risk

Questions on the Binomial Model
- What is the length of a period?
- Price can take more than two possible values.
- Trading takes place continuously.
If we let the period-length get smaller and smaller, we obtain the Black-Scholes option pricing formula:

\[
C(S, K, T) = SN\left( x \right) - KR^{-T}N\left( x - \sigma\sqrt{T} \right)
\]

where

- \( x \) is defined by

\[
x = \frac{\ln \left( \frac{S}{KR^{-T}} \right)}{\sigma\sqrt{T}} + \frac{1}{2} \sigma\sqrt{T}
\]

- \( T \) is in units of a year
- \( R = 1+r \), where \( r \) is the annual riskless interest rate
- \( \sigma \) is the volatility of annual returns on the underlying asset
- \( N(.) \) is the cumulative normal density function
An interpretation of the Black-Scholes formula:

- The call is equivalent to a levered long position in the stock.
- \( S \, N(x) \) is the amount invested in the stock
- \( K R^{-T} N \left( x - \sigma \sqrt{T} \right) \) is the dollar amount borrowed
- The option delta is \( N(x) = C_S \)
Example. Consider a European call option on a stock with the following data:
- S = 50, K = 50, T = 30 days
- The volatility $\sigma$ is 30% per year
- The current annual interest rate is 5.895%

Then

$$x = \frac{\ln \left( \frac{50}{50(1.05895)^{\frac{30}{365}}} \right)}{(0.3)\sqrt{\frac{30}{365}}} + \frac{1}{2}(0.3)\sqrt{\frac{30}{365}} = 0.0977$$

$$C = 50N(0.0977) - 50(1.05895)^{-\frac{30}{365}}N \left( 0.0977 - 0.3\sqrt{\frac{30}{365}} \right)$$
$$= 50(0.53890) - 50(0.99530)(0.50468)$$
$$= 1.83$$
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