Chapter 5
Consistency, Zero Stability, and the Dahlquist Equivalence Theorem

In Chapter 2 we discussed convergence of numerical methods and gave an experimental method for finding the rate of convergence (aka, global order of accuracy). When we devise a numerical method, however, we prefer to make a claim about the convergence properties before implementing it. In this chapter we will describe the conditions under which a numerical scheme can be claimed to be convergent.

18 Self-Assessment

Before reading this chapter, you may wish to review...
• stability [Unified S/S II Lecture 12 Notes]
• stability of difference equations [Wikipedia: Control Theory, Stability]
• basics of difference equations [6.003 Lecture 2 Slides]

After reading this chapter you should be able to...
• evaluate the consistency of a numerical method
• evaluate the zero stability of a numerical method
• describe the relationship between consistency, stability, and convergence

19 Zero stability

A numerical method is zero stable if the solution remains bounded as $\Delta t \to 0$ for finite final time $T$. Recall the general form for an $s$-step multi-step method as given in Definition 1:

$$v_{n+1} + \sum_{i=1}^{s} \alpha_i v_{n+1-i} = \Delta t \sum_{j=0}^{s} \beta_j f_{n+1-j}. \quad (34)$$

In the limit $\Delta t \to 0$ we have

$$v_{n+1} + \sum_{i=1}^{s} \alpha_i v_{n+1-i} = 0. \quad (35)$$

This recurrence relationship determines the characteristic, or unforced, behavior of the multi-step method. The method is zero stable if all solutions to (35) remain bounded.

**Definition 1 (Zero stability).** A multi-step method is zero stable if all solutions to

$$v_{n+1} + \sum_{i=1}^{s} \alpha_i v_{n+1-i} = 0$$
remain bounded as \( n \to \infty \).

To determine if a method is zero stable, we assume that the solution to the recurrence has the following form,

\[
v^n = v^0 z^n,
\]

where the superscript in the \( z^n \) term is in fact a power. Note: \( z \) can be a complex number. If the recurrence relationship has solutions with \( |z| > 1 \), then the multi-step method would be unstable. For our purposes, a multi-step method with a root of \( |z| = 1 \) is zero-stable provided the root is not repeated. (Note that in general, we need to be careful with the case of \( |z| = 1 \).)

Example 1. In Example 2, the most accurate two-step, explicit method was found to be,

\[
v^{n+1} + 4v^n - 5v^{n-1} = \Delta t \left( 4f^n + 2f^{n-1} \right).
\]

We will determine if this algorithm is stable. The recurrence relationship is,

\[
v^{n+1} + 4v^n - 5v^{n-1} = 0.
\]

Then, substitution of \( v^n = v^0 z^n \) gives,

\[
z^{n+1} + 4z^n - 5z^{n-1} = 0.
\]

Factoring this relationship gives,

\[
z^{n-1} (z^2 + 4z - 5) = z^{n-1} (z - 1) (z + 5) = 0.
\]

Thus, the recurrence relationship has roots at \( z = 1 \), \( z = -5 \), and \( z = 0 \) (\( n - 1 \) of these roots). The root at \( z = -5 \) will grow unbounded as \( n \) increases so this method is not stable.

To demonstrate the lack of convergence for this method (due to its lack of stability), we again consider the solution of \( u_t = -u^2 \) with \( u(0) = 1 \). These results are shown in Figure 1. These results clearly show the instability. Note that the solution oscillates as is expected since the large parasitic root is negative (\( z = -5 \)). Furthermore, decreasing \( \Delta t \) from 0.1 to 0.05 only causes the instability to manifest itself in shorter time (though the same number of steps). Clearly the method will not converge because of this instability.

## 20 Consistency

A numerical method is **consistent** if the multi-step discretization satisfies the governing equation in the limit \( \Delta t \to 0 \).

**Definition 2 (Consistency).** A numerical method is consistent if its local truncation error \( \tau \) has the property

\[
\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = 0.
\]

Let \( v^{n+1} = g(v^{n+1}, v^n, \ldots, t^{n+1}, t^n, \ldots) \) represent the numerical method as we have seen before. Then, if we write the ratio \( \tau / \Delta t \), we find

\[
\frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots) - u^{n+1}}{\Delta t},
\]

and substitute the Taylor series of \( u^{n+1} \),

\[
\frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots) - (u^n + \Delta tu^n + O(\Delta t^2))}{\Delta t}.
\]

We can rewrite this in terms of the slope of the numerical method

\[
\frac{\tau}{\Delta t} = \frac{g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots) - u^n - u^n + O(\Delta t))}{\Delta t}.
\]
If we now take the limit $\Delta t \to 0$, the last term vanishes and we find

$$\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - u^n - u^n.$$

It follows then that if the numerical method is consistent, we have $\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = 0$ and therefore

$$g(u^{n+1}, u^n, \ldots, t^{n+1}, t^n, \ldots, \Delta t) - u^n = u^n.$$

In other words, the multi-step discretization satisfies the governing equation in the limit as $\Delta t \to 0$.

**Thought Experiment** For consistency we require $\lim_{\Delta t \to 0} \tau / \Delta t = 0$. Why is it insufficient to require the local truncation error $\tau$ to vanish in the limit, $\lim_{\Delta t \to 0} \tau = 0$?

We can extend the definition of consistency directly to a criterion on the order of accuracy of the numerical method. Requiring $\lim_{\Delta t \to 0} \tau / \Delta t = 0$ implies that $\tau = O(\Delta t^{p+1})$ where $p \geq 1$ since then $\tau / \Delta t = O(\Delta t^p)$, which only vanishes in the limit if $p \geq 1$. This means that a consistent method must be at least first-order accurate.

**Exercise 1.** Which of the following numerical methods is consistent?
21 Dahlquist Equivalence Theorem

The Dahlquist Equivalence Theorem is the primary tool for assessing whether or not a numerical scheme is convergent. Using the concepts of consistency and zero stability alone, we can draw a conclusion about the convergence.

To summarize from above, we have the following concepts:

**Consistency:** In the limit \( \Delta t \to 0 \), the method gives a consistent discretization of the ordinary differential equation.

**Zero stability:** In the limit \( \Delta t \to 0 \), the method has no solutions that grow unbounded as \( N = T/\Delta t \to \infty \).

The Dahlquist Equivalence Theorem guarantees that a method that is consistent and stable is convergent, and also that a convergent method is consistent and stable:

**Theorem 1 (Dahlquist Equivalence Theorem).** A multi-step method is convergent if and only if it is consistent and stable.

**Exercise 2.** The numerical scheme defined by the equation \( v^{n+1} = \frac{1}{2} v^n + \frac{1}{2} \Delta t (f^n + f^{n-1}) \) is

(a) zero stable, not consistent, and convergent
(b) not zero stable, consistent, and not convergent
(c) zero stable, consistent, and convergent
(d) none of the above