16.901: Finite-difference PDE Project
The Impact of Film Cooling on Combustor Liner Temperatures
Sample Solution

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1 Background

The temperatures within the primary zone of a combustor are significantly higher than the temperatures which most materials can withstand without significant deterioration. Thus, a critical aspect of the design of a combustor is the development of a method to cool the liner walls of a combustor such that the temperatures which the liner temperatures are well below the limit of the material. A typical method to cool a combustor liner is through film cooling. Film cooling consists of diverting air from the main flow path prior to combustion and then reintroducing this air along the liner surface to provide a film of cooler air to protect the liner.

In this project, you will simulate the air flow and the liner wall to estimate the effectiveness of a film cooling strategy. The specific model problem is shown in Figure 1 and the values of the specific parameters are given in Table 1. We will assume that the flow enters the computational domain of interest at \( x = 0 \) with the following conditions:

For \( x = 0, y > h \) \( : U = U_{\text{hot}}, T = T_{\text{hot}}. \)
For \( x = 0, 0 < y < h \) \( : U = U_{\text{cool}}, T = T_{\text{cool}}. \)
For \( x = 0, -t_w \leq y \leq 0 \) \( : U = 0, T = T_{\text{cool}}. \) (Note: no convection in liner wall!)
For \( x = 0, y < -t_w \) \( : U = U_{\text{cool}}, T = T_{\text{cool}}. \)

The velocity throughout the domain is only in the x-direction and is given by the inlet value, i.e.,

\[
U(x,y) = U(0,y), \quad V(x,y) = 0.
\]

The governing equation for our model problem will be the convection-diffusion equation and we will include a variable thermal conductivity to handle the change in conductivity between the air flow and the liner wall. Thus, the governing equation throughout the entire domain of interest is,

\[
U \frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right).
\]

At the outlet of the computational domain, we will use a ‘parabolized’ version of this equation in which we will assume the second derivative in \( x \) is small compared to the second derivative in \( y \),

\[
At \ x = L, \quad U \frac{\partial T}{\partial x} = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right).
\]

2 Tasks

2.1 Finite Difference Approximations for Non-uniform Grid Spacing

To efficiently study the film cooling problem, a grid with small \( y \)-spacing is desired in the region around the liner but large \( y \)-spacing away from the liner. Thus, the \( y \)-spacing will be a function of the \( j \)-index,

\[
\Delta y_{j+\frac{1}{2}} \equiv y_{j+1} - y_j.
\]
Figure 1: Combustor liner with film cooling

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_d$</td>
<td>air conductivity</td>
<td>0.1 W/(m K)</td>
</tr>
<tr>
<td>$k_w$</td>
<td>wall conductivity</td>
<td>26.0 W/(m K)</td>
</tr>
<tr>
<td>$h$</td>
<td>height of cooling passage</td>
<td>0.003 m</td>
</tr>
<tr>
<td>$L$</td>
<td>axial length between cooling passages</td>
<td>0.3 m</td>
</tr>
<tr>
<td>$U_{hot}$</td>
<td>velocity of hot flow</td>
<td>100</td>
</tr>
<tr>
<td>$U_{cool}$</td>
<td>velocity of cool flow</td>
<td>150</td>
</tr>
<tr>
<td>$T_{hot}$</td>
<td>temperature of hot flow</td>
<td>2200 K</td>
</tr>
<tr>
<td>$T_{cool}$</td>
<td>temperature of cool flow</td>
<td>800 K</td>
</tr>
<tr>
<td>$t_w$</td>
<td>thickness of liner wall</td>
<td>0.0015 m</td>
</tr>
</tbody>
</table>

Table 1: Parameter definitions and values
In the x-direction, the spacing will be kept constant and equal to \( \Delta x \). Using a Taylor series analysis, we will find the values of \( a \) and \( b \) such that,

\[
a \left( k_{i+1} \frac{T_{i+1} - T_j}{\Delta y_{i+1}} \right) - b \left( k_{i-1} \frac{T_{i-1} - T_j}{\Delta y_{i-1}} \right) = \frac{1}{\Delta y} \left( \frac{\partial T}{\partial y} \right)_{j+1} - \frac{1}{\Delta y} \left( \frac{\partial T}{\partial y} \right)_{j-1}
\]

(1)
is an approximation to \( \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \) at the \( j \)-th location. Note, the notation \( k_{i+1} \) is defined as,

\[
k_{i+1} = k \left( \frac{1}{2} (y_j + y_{j+1}) \right).
\]

To begin this analysis, we note that a Taylor series analysis shows that,

\[
k_{i+1} \frac{T_{i+1} - T_j}{\Delta y_{i+1}} = k_{i+1} \frac{\partial T}{\partial y} \bigg|_{j+\frac{1}{2}} + \frac{1}{24} \Delta y_{i+1} k_{i+1} \frac{\partial^3 T}{\partial y^3} \bigg|_{j+\frac{1}{2}} + \text{H.O.T.}
\]

(2)

\[
k_{i-1} \frac{T_{i-1} - T_j}{\Delta y_{i-1}} = k_{i-1} \frac{\partial T}{\partial y} \bigg|_{j-\frac{1}{2}} + \frac{1}{24} \Delta y_{i-1} k_{i-1} \frac{\partial^3 T}{\partial y^3} \bigg|_{j-\frac{1}{2}} + \text{H.O.T.}
\]

(3)

where H.O.T. stands for higher order terms. Furthermore, we can expand the first and third derivative terms about node \( j \),

\[
k_{i+\frac{1}{2}} \frac{\partial T}{\partial y} \bigg|_{j+\frac{1}{2}} = \left( k \frac{\partial T}{\partial y} \right)_{ij} + \frac{1}{2} \Delta y_{i+\frac{1}{2}} \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right)_{ij} + \frac{1}{8} \Delta y^2_{i+\frac{1}{2}} \frac{\partial^2}{\partial y^2} \left( k \frac{\partial T}{\partial y} \right)_{ij} + \text{H.O.T.}
\]

(4)

\[
k_{i-\frac{1}{2}} \frac{\partial^3 T}{\partial y^3} \bigg|_{j+\frac{1}{2}} = \left( k \frac{\partial^3 T}{\partial y^3} \right)_{ij} + \text{H.O.T.}
\]

(5)

Substituting Equations (2), (3), (4), and (5) into Equation (1) gives the following result for each order,

-1 order terms

\[
= \frac{2(a-b)}{\Delta y_{i+\frac{1}{2}} + \Delta y_{i-\frac{1}{2}}} \left( k \frac{\partial T}{\partial y} \right)_{ij}
\]

0 order terms

\[
= \frac{a \Delta y_{i+\frac{1}{2}} + b \Delta y_{i-\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}} + \Delta y_{i-\frac{1}{2}}} \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right)_{ij}
\]

1 order terms

\[
= \frac{a \Delta y^2_{i+\frac{1}{2}} + b \Delta y^2_{i-\frac{1}{2}}}{\Delta y_{i+\frac{1}{2}} + \Delta y_{i-\frac{1}{2}}} \left( \frac{1}{4} \frac{\partial^2}{\partial y^2} \left( k \frac{\partial T}{\partial y} \right)_{ij} + \frac{1}{12} \left( k \frac{\partial^3 T}{\partial y^3} \right)_{ij} \right)
\]

The -1 order terms must be eliminated, so this requires that \( a = b \). Then, since the 0 order terms must result in \( \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) \), clearly \( a = b = 1 \). Thus, with these values for \( a \) and \( b \), the first order terms produce,

1 order terms

\[
= \left( \Delta y_{i+\frac{1}{2}} - \Delta y_{i-\frac{1}{2}} \right) \left( \frac{1}{4} \frac{\partial^2}{\partial y^2} \left( k \frac{\partial T}{\partial y} \right)_{ij} + \frac{1}{12} \left( k \frac{\partial^3 T}{\partial y^3} \right)_{ij} \right).
\]

Thus, the approximation is only first order accurate in general. But, if the change in the grid spacing is second order, that is if \( \Delta y_{i+\frac{1}{2}} - \Delta y_{i-\frac{1}{2}} = O(\Delta y^2_{i+\frac{1}{2}}) \) then the approximation will be second order. This places a requirement on the amount of stretching which a cell can undergo while still maintaining second-order accuracy.

### 2.2 Grid Generation

A constant spacing in the x-direction was used. In the y-direction, the algorithm used for grid generation in this study is as follows:
To simplify the implementation, we will require that the surfaces of the liner wall at \( y = -t_w \) and \( y = 0 \) are both grid lines.

Within the liner wall, a constant \( y \)-spacing was used, \( \Delta y_w \). Furthermore, from \( M_{lower} t_w \leq y \leq M_{upper} t_w \), this constant spacing was maintained where \( M_{lower} \) and \( M_{upper} \) are user-defined inputs. In this study, fixed values were used with \( M_{lower} = 3 \) and \( M_{upper} = 5 \).

Outside of this refined region, the grid was stretched using a geometric scaling, \( \Delta y_j + \frac{1}{s} = s \Delta y_{j-1} + \frac{1}{s} \), where \( s \) is a user-defined input parameter. A study of the effect of this stretching is given later in the report.

The farfield boundaries distances were set at \( H_{upper} \) and \( H_{lower} \). In reality, the grid outer boundary is required to lie at least at this distance but is generally larger since the geometric grid spacing growth did not permit an exact specification. A study of the impact of this parameter is given later in this report.

### 2.3 Implementation of Finite Difference Method for Analysis of Liner

A second-order upwind approximation for \( \frac{\partial T}{\partial x} \) is,

\[
\frac{\partial T}{\partial x} = \frac{\frac{3}{2}T_{j,k} - 2T_{j-1,k} + \frac{1}{2}T_{j-2,k}}{\Delta x},
\]

This approximation can be used throughout the domain except at \( j = 1 \) and \( j = 2 \). Since Dirichlet conditions are used at \( j = 1 \), this leaves only \( j = 2 \). At this location, a central-difference approximation is used. The \( y \)-differencing was described in Section 2.1. The final version of the code is available on the course website.

### 2.4 Accuracy Study of Temperature Predictions

#### 2.4.1 Basic resolution study

For these simulations the upper and lower boundaries were placed approximately at 20\( h \) above and 10\( h \) below the wall surfaces, respectively. The grid was resolved with constant \( y \)-spacing for 3\( t_w \) below the wall and 5\( t_w \) above the wall with the specific spacing given in the table. The grid was stretched at a rate of \( s = 1.5 \) outside these constant spacing regions. The variation of maximum wall temperature (in degrees K) with \( \Delta x \) and \( \Delta y_w \) is given in Table 2.

The results show that at the coarsest settings (i.e. \( \Delta x/L = 5 \) and/or \( \Delta y_w/t_w = 2 \)) a noticeable difference occurs from the finer mesh results. So, these resolutions are deemed unsuitable (though in some cases \( T_{max} \) is about the same as on the finest mesh tested). Also evident is that the \( x \)-spacing has less control on the accuracy as variations of \( T_{max} \) for \( \Delta x/L = 10, 20, \) and 30 are about one degree K. Also, the answer for \( \Delta y_w/t_w = 4 \) are noticeably low. Thus, the best grid which maximizes efficiency and at reasonable accuracy would seem to be \( \Delta x/L = 10 \) and \( \Delta y_w/t_w = 8 \). Figure 2 contains a plot of the upper surface wall temperature for three of the finer meshes.

#### 2.4.2 Impact of farfield boundary placement

Next, the impact of the farfield placement was studied. For the upper boundary, the distribution of outlet temperature and the upper surface wall temperature versus the upper boundary location is shown in Figure 3. The results show that for \( H_{upper} > 20h \) there is little dependence on the temperature profiles. Similar results were found for the lower boundary with the cut-off being \( H_{lower} > 15h \).

#### 2.4.3 Impact of stretching

The stretching was then studied with \( H_{upper} = 20h \) and \( H_{lower} = 15h \). The results show a relatively small dependence of \( T_{max} \) on the stretching on the order of 1-2 degrees K. Thus, a mid-range value of \( s = 1.5 \) was selected for the baseline grid.
Table 2: Variation of maximum wall temperature (in degrees K) with near wall y-spacing and x-spacing. For these simulations the upper and lower boundaries were placed approximately at 20h above and 10h below the wall surfaces, respectively. The grid was resolved with constant y-spacing for 3t_w below the wall and 5t_w above the wall with the specific spacing given in the table. Grid was stretched at a rate of s = 1.5 outside these constant spacing regions.

<table>
<thead>
<tr>
<th>Δx/L</th>
<th>Δy_w/t_w</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>1280.9</td>
</tr>
<tr>
<td>10</td>
<td>1276.2</td>
</tr>
<tr>
<td>20</td>
<td>1277.0</td>
</tr>
<tr>
<td>30</td>
<td>1277.9</td>
</tr>
</tbody>
</table>

Figure 2: Impact of grid resolution on upper wall surface temperature (in degrees K) for (Δx/L, Δy_w/t_w) = (10, 4), (20, 8), and (30, 12) with line types dash-dot, dashed, and solid, respectively.
Figure 3: Variation of outlet and upper surface wall temperature with farfield boundary location. $H_{upper} = 10h$ (solid), $20h$ (dashed), and $30h$ (dash-dot).

Figure 4: Variation of maximum wall temperature with $y$-stretching factor.
2.4.4 Baseline grid characteristics and solution plots

From the above accuracy studies, the parameters for the baseline grid are given in Table 3. Plots of the contours of the temperature in the field are shown in Figure 5. As predicted on this mesh, the maximum wall temperature is approximately 1286.6 degrees K with an estimated error of about ±5 degrees K.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{upper}$</td>
<td>Location of upper boundary</td>
<td>20h</td>
</tr>
<tr>
<td>$H_{lower}$</td>
<td>Location of lower boundary</td>
<td>15h</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>x-spacing</td>
<td>$L/10$</td>
</tr>
<tr>
<td>$\Delta y_w$</td>
<td>Fine grid (near wall) y-spacing</td>
<td>$t_w/8$</td>
</tr>
<tr>
<td>$y_{upper}$</td>
<td>Region above wall with fine grid</td>
<td>5h</td>
</tr>
<tr>
<td>$y_{lower}$</td>
<td>Region below wall with fine grid</td>
<td>3h</td>
</tr>
<tr>
<td>$s$</td>
<td>Y-stretching factor</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Table 3: Parameters for baseline grid. Note: for these setting, the grid size is $11 \times 10^3$.

![Temperature contours](image)

Figure 5: Temperature contour distribution for baseline grid simulation. Maximum wall temperature is 1286.38 degrees K.

2.5 Von Neumann Analysis

Suppose that the model were to be used for an unsteady problem, so that the governing equation became,

$$\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial x} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right).$$
We will perform on Von Neumann analysis on the linearized version of this equation applying the spatial
discretization from above with a Forward Euler time integration. To do the analysis, we also need to assume
a constant spacing in $y$. Specifically, the semi-discrete form of the model equation for this problem is,
\[
\frac{\partial T_{j,k}}{\partial t} + U \frac{4}{\Delta x} T_{j,k} - 2 T_{j-1,k} + \frac{k}{\Delta x^2} T_{j-2,k} = k \left[ \frac{T_{j+1,k} - 2 T_{j,k} + T_{j-1,k}}{\Delta x^2} \right] + k \left[ \frac{T_{j+1,k+1} - 2 T_{j,k} + T_{j-1,k+1}}{\Delta y^2} \right]
\]
Following the usual Von Neumann analysis, we then make the following substitution for $T_{j,k}$,
\[T_{j,k} = \hat{g}(\beta_x, \beta_y) e^{i(\beta_x j + k \beta_y)},\]
where $\beta_x$ and $\beta_y$ are the discrete (non-dimensional wavenumbers) and $\hat{g}(t, \beta_x, \beta_y)$ is the amplification factor
for the waves. This results in the following governing equation for $\hat{g}$,
\[
\frac{\partial \hat{g}}{\partial t} = \lambda(\beta_x, \beta_y) \hat{g},
\]
where,
\[
\lambda \Delta t = - \frac{U \Delta t}{\Delta x} \left( \frac{3}{2} - 2 e^{-i \beta_x} + \frac{1}{2} - 2 i \beta_x \right) + 2 \frac{k \Delta t}{\Delta x^2} (\cos \beta_x - 1) + \frac{k \Delta t}{\Delta y^2} (\cos \beta_y - 1)
\]

### 2.5.1 Liner wall stability constraint

In the liner wall, the following values are found for the parameters in $\lambda$ using the baseline grid,
\[
\frac{U}{\Delta x} = 0, \quad \frac{k}{\Delta x^2} = 2.9 \times 10^4/sec, \quad \frac{k}{\Delta y^2} = 7.4 \times 10^9/sec,
\]
Note that the large aspect ratio in the wall, $\Delta x/\Delta y = 160$ results in a value of $k/\Delta y^2$ which is four orders of
magnitude larger than $k/\Delta x^2$. Since $U = 0$ in the wall, the eigenvalues are purely real (and non-positive).
To remain in the stability region for Forward Euler, the negative extent of these eigenvalues must be no
larger than 2 in magnitude. This requires that $\Delta t = 6.7 \times 10^{-10}$ seconds which is a very severe timestep
restriction when compared to the time for the hot gas to convect across the liner, i.e., $L/U_{hot} = 2 \times 10^{-3}$
seconds. Thus, $\Delta t L/U_{hot} = 3.35 \times 10^{-7}$. A plot of the eigenvalues is shown in Figure 6.

### 2.5.2 Near wall stability constraint

Near the liner wall though in the gas, the following values are found for the parameters in $\lambda$,
\[
\frac{U}{\Delta x} = 5.0 \times 10^3/sec, \quad \frac{k}{\Delta x^2} = 1.1 \times 10^2/sec, \quad \frac{k}{\Delta y^2} = 2.8 \times 10^6/sec,
\]
As before, the large aspect ratio near the wall causes a severe timestep restriction. However, since the
gas conductivity is two orders less than the wall conductivity, the timestep restriction is not as severe as
in the liner wall. In this case, we also have a convection velocity which will give rise to imaginary parts
for $\lambda$. However, since $U/\Delta x$ is much smaller than $k/\Delta y^2$, the resulting eigenvalues are tightly
cluster about the real axis. For this case, the timestep restriction is approximately, $\Delta t = 1.77 \times 10^{-7}$ seconds or
$\Delta t L/U_{hot} = 8.85 \times 10^{-5}$. A plot of the eigenvalues is shown in Figure 7.

### 2.5.3 Farfield stability constraint

Near the farfield boundaries, the following values are found for the parameters in $\lambda$,
\[
\frac{U}{\Delta x} = 3.3 \times 10^3/sec, \quad \frac{k}{\Delta x^2} = 1.1 \times 10^2/sec, \quad \frac{k}{\Delta y^2} = 2.5 \times 10^2/sec,
\]
In the farfield, the $y$-spacing is much larger thus $k/\Delta y^2$ is no longer a severe limiting factor in the stability. In
fact, the convection scale is about an order of magnitude larger. In this case, however, the stability constraint
is a little more subtle. Specifically, consider Figure 8 which shows the eigenvalues for $\Delta t = 1.1 \times 10^{-4}$ seconds.
Figure 6: Eigenvalue plots for liner wall conditions with $\Delta t = 6.7 \times 10^{-10}$ sec. Stability boundary of the Forward Euler method is overlayed.

Figure 7: Eigenvalue plots for near wall conditions with $\Delta t = 1.77 \times 10^{-7}$ sec. Stability boundary of the Forward Euler method is overlayed.
As is highlighted in the figure, a few eigenvalues lie just outside the stability boundary for the Forward Euler. In order to place them inside the stability region, a much smaller \( \Delta t \) is needed than might be expected from the CFL constraint. Specifically, through trial-and-error testing, it was found that the maximum \( \Delta t = 2.0 \times 10^{-5} \) seconds or \( \Delta t / U_{\text{hot}} = 1 \times 10^{-2} \). A plot of the eigenvalues is shown in Figure 9. This translates into a CFL number of \( U_{\text{hot}} \Delta t / \Delta x = 0.0667 \) which is 30 times less than the value of two required from the CFL condition.

![Eigenvalue plot](image)

**Figure 8:** Eigenvalue plots for farfield conditions with \( \Delta t = 1.1 \times 10^{-4} \) sec. Stability boundary of the Forward Euler method is overlayed.
Figure 9: Eigenvalue plots for farfield conditions with $\Delta t = 2.0 \times 10^{-5}$ sec. Stability boundary of the Forward Euler method is overlayed.