1. **Control of Hydraulic Servomechanism.** We return to the Hydraulic Servomechanism of Problem 1 in Homework Assignment 6 with additional data which permits quantitative treatment of the design requirements. In Eq.(5) of the solution for Problem 1 of Assignment 6 the transfer function from current \( i \) to table angular velocity \( \omega_t \) was derived as

\[
\frac{\omega_t(s)}{i(s)} = \frac{P_{\text{max}} DN}{i_{\text{max}} I_t s + \lambda_i}
\]

(1)

In Assignment 6, data was provided which enabled us to infer that \( \lambda_i = 2\pi \) rad/sec. Now we are given that a steady current of 1 amp produces a steady angular velocity of 30 RPM, or \( \pi \) rad/sec. The steady-state ratio of angular velocity to current is the zero-frequency value of the frequency response of the angular velocity to current excitation which is the same as the transfer function (1) evaluated at \( s = 0 \), so that

\[
\frac{\omega_t(0)}{i(0)} = \frac{\pi}{1} = \frac{P_{\text{max}} DN}{i_{\text{max}} I_t 2\pi} \quad \text{or} \quad \frac{P_{\text{max}} DN}{i_{\text{max}} I_t} = 2\pi^2
\]

(2)

When (2) is substituted back into (1) the transfer function from current to table angular velocity takes the more compact form

\[
\frac{\omega_t(s)}{i(s)} = \frac{2\pi^2}{s + 2\pi} \text{ rad/sec per amp}
\]

(3)

and the open-loop transfer function for the system with proportional control shown in Fig. 1 is (See Eq.(9) in the Solution of Homework Assignment 6)

\[
OLT F(s) = \frac{e_{\text{sensor}}(s) \omega_t(s)}{i(s) e_{\text{error}}(s)} = GM \frac{2\pi^2}{s + 2\pi} \frac{1}{s}
\]

(4)

(a) Consider the closed-loop system which produces the sensor voltage as output from the reference voltage as input. Its transfer function can be derived from the OLT F(s) above by writing

\[
e_{\text{sensor}} = OLT F(s) e_{\text{error}} = OLT F(s)(e_{\text{ref}} - e_{\text{sensor}})
\]

and solving for

\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{OLT F(s)}{1 + OLT F(s)} = \frac{2\pi^2 GM}{s^2 + 2\pi s + 2\pi^2 GM}
\]

(5)

Note that this has the standard form

\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]
with

\[ \omega_n = \pi \sqrt{2GM} \quad \text{and} \quad 2\zeta \omega_n = 2\pi \quad \text{or} \quad \zeta = \frac{1}{\sqrt{2GM}} \]

The eigenvalues (roots of the denominator) are

\[ \lambda_1, \lambda_2 = -\zeta \omega_n \pm j\omega_d = -\pi \pm j\pi \sqrt{2GM - 1} \]

When the gain G is varied the values of \( \omega_n \) and \( \zeta \) change, but their product \( \zeta \omega_n \) remains constant. As \( GM \) varies from 0.5 amps to \( \infty \) the eigenvalues move up and down the line \( s = -\pi \). The decay rate for the envelope of under-damped responses is \( \zeta \omega_n \).

(i) When \( 0 \leq \zeta < 1.0 \) the step response has overshoot. When \( 1.0 \leq \zeta \leq \infty \) there is no overshoot. The response with the fastest settling time (and no overshoot) is the critically damped response with \( \zeta = 1.0 \). Taking into account that \( M = 1.0 \) volt/radian, the magnitude of the gain G required to produce \( \zeta = 1.0 \) is \( G = 0.5 \) amp/volt. The critically damped response to a unit step of \( e_{ref} \) applied at \( t = 0 \) is

\[ e_{sensor} = 1.0 - \exp\{-\omega_n t\}(1 + \omega_n t) \]

which reaches the value 0.98 when \( t_{\text{settling}} = 1.856 \) sec.

(ii) When \( 0 \leq \zeta < 1.0 \) the under-damped response to a unit step input at \( t = 0 \) is

\[ e_{sensor} = 1.0 - \exp\{-\omega_n t\}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right) \]

The peak value of this response, at \( t_{\text{peak}} = \pi/\omega_d \),

\[ e_{\text{peak}} = 1.0 + \exp\{-\pi \frac{\zeta}{\sqrt{1-\zeta^2}}\} \]

is a function only of the damping ratio \( \zeta \). The peak value is 1.05 (5 % overshoot) when \( \zeta = 0.690 \) which requires that \( 1/\sqrt{2GM} = 0.690 \), or, with \( M = 1.0 \) volt/radian, \( G = 1.050 \) amp/volt. The response (5) passes downward through the value 1.02 when \( t_{\text{settling}} = 1.317 \) sec.
(b) The poles associated with the critically damped case \((G = 0.5)\) are located at \(s = -\pi\) and the poles associated with the 5% overshoot case \((G = 1.050)\) are located at \(s = -\pi \pm j3.30\). These pole locations are sketched in Fig. 2. Higher gain values produce poles farther out on the vertical line \(s = -\pi\).

![Figure 2: Pole Locations for Proportional Controller.](image)

(c) Transient response performance will not be improved by further increases in gain. The poles for the 5% overshoot case are just about on the 45 degree borderline of desirable performance. Larger gains will decrease the value of \(\zeta\) and therefore increase the overshoot without changing the settling time appreciably. The envelope of all under-damped responses decays at the rate \(\zeta \omega_n = \pi\) independently of the gain.

(d) In between the gains of \(G = 0.5\) and \(G = 1.05\) examined in detail above, is a range of gains with overshoots between zero and 5%, and with settling times between 1.856 sec and 1.317 sec. Within these somewhat narrow ranges there is some opportunity for trading off a decrease in overshoot against an increase in settling time with this controller.

With the PD controller \(i(s) = G(1 + sK_d)(\epsilon_{error})\) the transfer function which had been

\[
\frac{i(s)}{\epsilon_{error}} = G
\]
in Eq.(4) is replaced by
\[
\frac{i(s)}{\varepsilon_{\text{error}}} = G(1 + sK_d)
\]
and the closed-loop transfer function of Eq.(5) is replaced by
\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{2\pi^2GM(1 + sK_d)}{s^2 + 2\pi s + 2\pi^2GM(1 + sK_d)}
\]
(6)

(e) To place the zero in (6) at 1.1 Hz = 6.9115 rad/sec, take \(K_d = 0.1447\) sec and get
\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{2.856GMs + 2\pi^2GM}{s^2 + s(2\pi + 2.856GM) + 2\pi^2GM}
\]
(7)

(f) To use MATLAB to display the step response and the pole-zero map for various gains, it is necessary to evaluate (7) numerically for each gain-value. In addition to the three cases asked for in the problem statement, we also include for comparison the step response and pole map for the 5% overshoot proportional controller with the closed-loop transfer function (5)
\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{2\pi^2GM}{s^2 + 2\pi s + 2\pi^2GM} = \frac{20.7}{s^2 + 6.28s + 20.7}
\]
To get the numerical values we have used \(M = 1.0\) volts/radian and \(G = 1.05\) amps/volt.

(i) The closed-loop transfer function for the P-D controller (7) has the following numerical form when \(GM = 0.5\) amps/radian
\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{1.428s + 9.87}{s^2 + 7.71s + 9.87}
\]
(ii) When \(GM = 1.05\) amps/radian, the numerical form of (7) is
\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{3.00s + 20.7}{s^2 + 9.28s + 20.7}
\]
(iii) Finally, when \(GM = 2 \times 1.05 = 2.1\) amps/radian, the numerical form of (7) is
\[
\frac{e_{\text{sensor}}(s)}{e_{\text{ref}}(s)} = \frac{6.00s + 41.4}{s^2 + 12.28s + 41.4}
\]
The desired step responses and pole-zero maps are then obtained from the following MATLAB session.

```matlab
sysP = tf(20.7, [1 6.28 20.7]);
sysPDi = tf([1.428 9.87], [1 7.71 9.87]);
```
sysPDii = tf([3.0 20.7], [1 9.28 20.7]);

sysPDiii = tf([6.0 41.4], [1 12.28 41.4]);

subplot(2,2,1), step(sysP)
subplot(2,2,2), step(sysPDi)
subplot(2,2,3), step(sysPDii)
subplot(2,2,4), step(sysPDiii)

subplot(2,2,1), pzmap(sysP)
subplot(2,2,2), pzmap(sysPDi)
subplot(2,2,3), pzmap(sysPDii)
subplot(2,2,4), pzmap(sysPDiii)

MATLAB displays the four cases in a single figure according to the following plan

<table>
<thead>
<tr>
<th>Proportional</th>
<th>P-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 % O.S.</td>
<td>(i)</td>
</tr>
<tr>
<td>$G = 1.05$</td>
<td>$G = 0.5$</td>
</tr>
</tbody>
</table>

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P-D</td>
<td>P-D</td>
</tr>
<tr>
<td>(ii)</td>
<td>(iii)</td>
</tr>
<tr>
<td>$G = 1.05$</td>
<td>$G = 2.1$</td>
</tr>
</tbody>
</table>

Figure 3: MATLAB plot layout.

The step responses are shown in Fig. 4 and the pole-zero maps are shown in Fig. 5
Figure 4: Comparison of Unit Step Responses.

(g) In case (i) the response is dominated by the single slow pole at $s = -1.62$ rad/sec. The response is essentially a first-order response. In case (ii) the two real poles and the zero are fairly close together and all must be considered to determine the response. In case (iii) the pair of conjugate complex poles are the only poles and thus may be considered a dominant pole pair. They have a real part that is quite close to the zero at $s = -6.91$ rad/sec, so the response is influenced by both the poles and the zero.

(h) Comparing the step responses in Fig. 4, we see that case (i) looks like a first-order response, with no overshoot but much slower rise-time and much longer settling-time than the response obtained by simple proportional control with 5% overshoot. In case (ii), with slightly more than twice the gain as in case (i), there is still no overshoot but the rise-time and the settling-time are comparable with those for the proportional controller with 5% overshoot. In case (iii), with twice the gain of case (ii), there is a miniscule overshoot (less than 0.02%), but now the rise-time and the settling-time are only about half that
for the proportional controller with 5% overshoot. It appears that larger gains could decrease the rise-time and settling-time even further, without introducing significant amounts of overshoot.

2. **Actuated Machine Tool Head.** We return to the Piezo-Electric Tool Actuator which appeared on the 2.004 Final Exam last June (See 2.004 Spring 2000 Solution of Final Examination). The solution here follows the Notes “Models of Piezo-Electric Transducers” distributed this week. The physical system consists of a mass \( m \) which can move in the \( x \)-direction, positive to the right, acted on by an elastic restraining force, \(-kx\), a damping force, \(-bx\), and a piezo-electric force

\[
f = dk_{sc}e_a - k_{sc}x
\]

where \( e_a \) is the voltage applied to the transducer, \( d \) is the given piezo-electric strain constant and \( k_{sc} \) is the short-circuit stiffness of the transducer, which must be determined from the given information regarding the open-circuit stiffness and the free capacitance.
(a) The equation of motion is

\[ dk_{sc} e_a - k_{sc} x - k x - b \dot{x} = m \ddot{x} \]

and the transfer function from voltage to displacement is

\[ P(s) = \frac{x(s)}{e_a(s)} = \frac{dk_{sc}}{ms^2 + bs + (k + k_{sc})} \] (1)

The equation of motion is a second-order differential equation and the transfer function has a second-order polynomial in the denominator, both of which indicate a second-order dynamic model.

Before going further, let us determine the value of the short-circuit stiffness \( k_{sc} \).

The open-circuit stiffness is

\[ k_{oc} = \frac{A}{h} Y = \frac{(0.01)^2}{0.001} 7.1 e 10 = 7.1 e 9 \text{ N/m} \]

and the free capacitance is

\[ C_{fr} = \frac{A}{h} \varepsilon_0 \varepsilon_r = \frac{(0.01)^2}{0.001} (8.85 e - 12)(450) = 3.98 e - 10 \text{ Farads} \]

Now on page 4 of the Notes “Models of Piezo-Electric Transducers” is the following formula for \( k_{sc} \)

\[ k_{sc} = \frac{1}{k_{oc} + \frac{d^2}{C_{fr}}} = \frac{k_{oc}}{1 + \frac{d^2 k_{oc}}{C_{fr}}} \]

which on substitution of the above values of \( k_{oc} \) and \( C_{fr} \) along with the given value of \( d \), yields \( k_{sc} = 5.26 e 9 \text{ N/m} \). The numerical form of (1) is then

\[ P(s) = \frac{x(s)}{e_a(s)} = \frac{0.737}{2s^2 + 4e4s + 5.27 e 9} = \frac{0.368}{s^2 + 2e4s + 2.64 e 9} \] (2)

The behavioral parameters of the transducer are

\[ \omega_n = \sqrt{2.64 e 9} = 51, 400 \text{ rad/sec} \quad \text{or} \quad f_n = 8, 180 \text{ Hz} \]

and

\[ \zeta = \frac{2 \zeta \omega_n}{2 \omega_n} = \frac{2 e 4}{2 \times 51, 400} = 0.1946 \]

Note that this is an extremely stiff system, with an ultrasonic resonant frequency.
(b) The open-loop transfer function for the proportional control of Fig.6 is

\[ OLTF(S) = \frac{e_x(s)}{e_{error}} = GP(s)M_x = \frac{0.368GM_x}{s^2 + 2e4s + 2.64e9} \]

and thus the error transfer function is

\[ ERTF(s) = \frac{e_{error}}{e_r} = \frac{1}{1 + OLTF(s)} = \frac{s^2 + 2e4s + 2.64e9}{s^2 + 2e4s + 2.64e9 + 0.368GM_x} \]

The steady-state unit step response error is the limit as \( s \to 0 \) of \( ERTF(s) \). This steady-state error is

\[ e_{ss} = \frac{2.64e9}{2.64e9 + 0.368GM_x} \]

which is never zero for finite gain, although \( e_{ss} \) can be made small by making \( GM_x \) very large.

(c) The steady-state unit step response error \( e_{ss} \) can be reduced to 0.20 by taking

\[ GM_x = \frac{2.64e9}{0.368} \left( \frac{1}{0.20} - 1 \right) = 2.87e10 \text{ volt/m} \]

In practice such a large factor would be split into a very sensitive sensor with \( M_x \) of the order of one volt per 10 microns and with \( G \) of the order of 300,000.

(d) With PI control the simple gain \( G \) is replaced by the operator \( G(s + K_i)/s \). The open-loop transfer function then becomes

\[ OLTF(S) = \frac{e_x(s)}{e_{error}} = G^2s + \frac{K_i}{s} P(s)M_x = \frac{0.368GM_x(s + K_i)}{s(s^2 + 2e4s + 2.64e9)} \]

and the error transfer function becomes

\[ ERTF(s) = \frac{e_{error}}{e_r} = \frac{s(s^2 + 2e4s + 2.64e9)}{s(s^2 + 2e4s + 2.64e9) + 0.368GM_x(s + K_i)} \]

Here we see that the limit as \( s \to 0 \) of \( ERTF(s) \) is zero, so the steady-state step response error is zero.
(e) the closed-loop transfer function for the system with PI control is

\[
CLTF(s) = \frac{e_x(s)}{e_r(s)} = \frac{0.368GM_x(s + K_i)}{s(s^2 + 2e4 + 2.64e9) + 0.368GM_x(s + K_i)}
\]

(3)

The real part of the physical system poles is \(-\zeta \omega_n = -10,000\) rad/sec. If the zero is to be placed midway between \(s = 0\) and \(s = -10,000\), it is necessary to take \(K_i = 5,000\) rad/sec.

(f) With \(K_i = 5,000\) rad/sec and \(GM_x = 2.87e10\) volt/m, the closed-loop transfer function (3) becomes

\[
CLTF(s) = \frac{e_x(s)}{e_r(s)} = \frac{1.056e10s + 5.28e13}{s^3 + 2e4s^2 + 1.320e10s + 5.28e13}
\]

Figure 7: Step Response of Piezo-electric Transducer with PI Control.

The unit step response and the pole-zero map for this system is obtained from the following MATLAB session:
sysPE = tf([1.056E10 5.28E13], [1 2E4 1.320E10 5.28E13]);
step(sysPE)
pzmap(sysPE)

The resulting plot of the step response is shown in Fig. 7, and the resulting pole-zero map is shown in Fig. 8.

Figure 8: Pole-Zero Map for Piezo-electric Transducer with PI Control.

Note the high-frequency ringing (from the pair of complex-conjugate poles) superposed on the first-order step response (due to the real pole) in the step response displayed in Fig. 7.