1. **Drive mechanism for radial arm carrying read-write head.** The given Bode pots for the position of the arm in response to the motor voltage input show three segments: for low frequencies the magnitude drops 20 dB per decade with a phase lag of about 90 degrees, for intermediate frequencies the magnitude drops 40 dB per decade with a phase lag of about 180 degrees, and for high frequencies the magnitude drops 60 dB per decade with a phase lag of about 270 degrees. The break points between these segments are at 10 rad/sec and 10,000 rad/sec. This suggests a transfer function with three poles and no zeros. The sketch below shows an iconic model of the system.

The equations of motion are obtained by applying Kirchhoff’s voltage law to the electric side, and the angular momentum principle to the mechanical side. Let the total moment of inertia be denoted by

\[ I_{tot} = I_m + I_{arm} \]

**(Study the motion.** The motion here is simple rotation about a fixed axis. The kinematic equation is

\[ \frac{d\theta}{dt} = \omega \]

**(Constitutive equations.** In addition to the motor relations \( e_m = K_e \omega \) and \( \tau_m = K_T i \), shown on the sketch, there are the electrical relations

\[ e_R = iR \quad \text{and} \quad e_L = L \frac{di}{dt} \]
and the mechanical relation

\[ \tau_{fric} = B_m \omega \]

*Study the forces (torques, voltages).* The total torque available to increase the angular momentum of \( I_{tot} \) is

\[ \tau_{tot} = K_T i - \tau_{fric} \]

and the total voltage drop facing the input voltage is

\[ e_{tot} = RI + K_e \omega + L \frac{di}{dt} \]

Assembling these equations, the angular momentum equation is

\[ K_T i - B_m \omega = \frac{d}{dt} (I_{tot} \omega) \]

and Kirchhoff’s voltage law is

\[ e_{in} = RI + K_e \omega + L \frac{di}{dt} \]

(a) The previous two equations and the kinematic equation are three first-order differential equations for the variables \( i \), \( \omega \), and \( \theta \). The variables \( i(t) \) and \( \omega(t) \) can be eliminated to provide a single third-order differential equation for \( \theta(t) \). This kind of algebra is simplified if the equations are transformed to the \( s \)-domain.

\[
\begin{align*}
\omega(s) &= s \theta(s) \\
K_T i(s) &= (s I_{tot} + B_m) \omega(s) \\
e_{in}(s) &= (R + sL)i(s) + K_e \omega(s)
\end{align*}
\]

When \( \omega(s) \) and \( i(s) \) are eliminated, the resulting equation for \( \theta(s) \) can be written as a transfer function from the input voltage to the arm position

\[
\frac{\theta(s)}{e_{in}(s)} = \frac{K_T}{s[L I_{tot} \Omega^2 + (I_{tot} R + B_m L) \Omega + B_m R + K_T K_e]}
\]

This transfer function has no zeros and three poles.

(b) The model elements included are the two motor constants \( K_T \) and \( K_e \), the motor resistance \( R \) and the motor inductance \( L \), the total moment of inertia of the rotating parts \( I_{tot} \), and the motor friction parameter \( B_m \).

(c) The steady-state frequency response function for the ratio of the complex amplitude \( A_\theta \) of the position to the complex amplitude \( A_{e_{in}} \) of the input voltage is obtained from the transfer function by setting \( s = j\Omega \)

\[
\frac{A_\theta(\Omega)}{A_{e_{in}}(\Omega)} = \frac{K_T}{j\Omega[-L I_{tot} \Omega^2 + j(I_{tot} R + B_m L) \Omega + B_m R + K_T K_e]}
\]
The low-frequency asymptote of the frequency response is

$$\lim_{\Omega \to 0} \frac{A_\theta(\Omega)}{A_{e_{in}}(\Omega)} = \frac{K_T}{j\Omega(B_mR + K_cK_T)}$$

which decays at a rate of 20 dB per decade and has a phase lag of 90 degrees. The high-frequency asymptote of the frequency response is

$$\lim_{\Omega \to \infty} \frac{A_\theta(\Omega)}{A_{e_{in}}(\Omega)} = \frac{K_T}{-jLI_{tot}\Omega^3}$$

which decays at a rate of 60 dB per decade and has a phase lag of 270 degrees. These agree with the given frequency response.

A simpler motor model in which the inductance $L$ is neglected is sometimes employed. Such a model leads to the frequency response

$$\frac{A_\theta(\Omega)}{A_{e_{in}}(\Omega)} = \frac{K_T}{j\Omega[+jI_{tot}\Omega + B_mR + K_TK_c]}$$

which has the same low-frequency asymptote as the previous model but which only decays at a rate of 40 dB per decade for high frequencies. This simpler model does not provide an adequate model for the measured data.

2. **Maximum gain for stable control of a machine tool head.** An iconic model of the control system is shown in the sketch.

![Control system sketch](image)

The transfer function for the measuring system $M(s)$ is

$$\frac{e_x(s)}{x(s)} = \frac{Ea}{s + a}$$
(a) When the given differential equation for \( x(t) \) is transformed to the \( s \)-domain the transfer function \( P(s) \) is obtained

\[
P(s) = \frac{x(s)}{f(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{m(s^2 + 2\zeta\omega_n s + \omega_n^2)}
\]

where

\[
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100,000}{10}} = 100 \text{ rad/sec}
\]

and

\[
\zeta = \frac{b}{2m\omega_n} = \frac{1400}{2(10)(100)} = 0.70
\]

(b) The transfer function from actuator voltage input to displacement output is

\[
\frac{x(s)}{e_a(s)} = DP(s) = \frac{D}{ms^2 + bs + k} = \frac{D}{m(s^2 + 2\zeta\omega_n s + \omega_n^2)}
\]

The corresponding frequency response is obtained by substituting \( s = j\Omega \)

\[
\frac{A_x(\Omega)}{A_{e_a}(\Omega)} = \frac{D}{m(\omega_n^2 + 2j\zeta\omega_n\Omega - \Omega^2)} = \frac{1}{10(10,000 + 140j\Omega - \Omega^2)}
\]

A rough sketch of the Bode plots for this frequency response is sketched below.
(c) The frequency response of the measurement system is obtained by substituting \( s = j\Omega \) in the transfer function \( M(s) \)

\[
\frac{A_x(\Omega)}{A_x(\Omega)} = \frac{E_a}{a + j\Omega} = \frac{(1000)(10,000)}{10,000 + j\Omega}
\]

A rough sketch of the Bode plots for the frequency response of the measurement system is shown below.

The stability margin for the closed-loop system can be determined from the Bode plots for the open-loop system. In the present case the transfer function for the open-loop system is

\[
C(s)P(s)M(s) = G \frac{D}{m(s^2 + 2\zeta \omega_n s + \omega_n^2)} s + a
\]

and the corresponding frequency response is

\[
G \frac{D}{m(\omega_n^2 + 2j\zeta \omega_n \Omega - \Omega^2)} a + j\Omega
\]

The maximum gain for stable operation of the closed-loop system can be obtained by setting \( G \) equal to unity in the above expression and then determining the gain margin of the resulting frequency response. But the resulting frequency response is just the product of the two frequency responses for which the Bode plots have already been sketched. The Bode plots for the resulting frequency response is obtained by graphically adding the two previous Bode plots, as shown below. The gain margin, read off the graph is approximately 85 dB at a frequency of approximately 1500 rad/sec. This means that the borderline between stability and instability is not reached until 20 \( \log_{10} G \approx 85 \) dB, or \( G \approx 18000 \).
Gain Margin

Asymptotes for $M(s)$

Asymptotes for $C(s)P(s)$

Asymptotes for $C(s)P(s)M(s)$

Magnitude, dB

Phase, Degrees

[Graph showing Bode plots for $M(s)$, $C(s)P(s)$, and $C(s)P(s)M(s)$ with magnitude in dB and phase in degrees.]