1. Assume that a consumer's utility function is given by:

\[ u(x_1, x_2, x_3) = \beta_1 \log (x_1 - a_1) + \beta_2 \log (x_2 - a_2) + \beta_3 \log x_3 \]

and that the consumer faces consumer prices \( q_1 \) and \( q_2 \), with the price of good three normalized to unity. The consumer's endowment \( y \) is measured in units of good three.

(a) Find the indirect utility function and expenditure function corresponding to this set of preferences. (This is the famous Stone-Geary utility function, the basis for the linear expenditure system in demand analysis.)

(b) Use your results from (a) to find an analytic expression for the compensating variation. In the special case of \( a_1 = a_2 = .50 \) and \( \beta_1 = \beta_2 = .40 \), find the CV associated with a tax reform that changes \((q_1, q_2, y)\) from \((1, 1, 5)\) to \((2.0, 1.5, 5)\). How would you expect the EV to compare with the CV in this case (you do not need to compute the EV - just note whether you believe it would be larger or smaller, and why).

(c) Assuming that the differences between the pre- and post-reform consumer prices in (b) are due only to tax changes, i.e. that producer prices are fixed, compute the government's revenue from the new tax policy.

2. Consider another economy with two types of individuals. The first type has no choice regarding labor supply: he must supply one unit of labor at a wage of 10. The second class of individuals, with wages of 20 per unit labor, can either supply one-half unit (part-time work) or a full unit of labor. The utility functions for the two households are identical:

\[ u^i = y^i - t^i(y^i) + \log (1 - \beta L) \]

where \( y^i \) denotes wage income, \( L \) is labor supply, and \( t^i \) is the household's income tax payment. Note that there are only two possible values of \( y^i \) in this economy, so the tax function need only be defined at these points.

The government seeks to maximize a Rawlsian social welfare function:

\[ W = \min (u^1, u^2) \]

using an income tax. Assuming 25\% of the population is of type 1, and setting \( \beta = .95 \), find the income tax schedule which maximizes social welfare. Contrast this schedule with that when \( \beta = .50 \). Can you provide an intuition for the change?
3. Consider an economy with two types of individuals. Those who are in the primary labor market (P) inelastically supply one unit of labor at a market wage of 1. Those who are in the secondary labor market (S) earn a market wage of 0.5 per unit of labor, and they can choose to supply one unit of labor, one half unit of labor, or no labor. A household consists of two individuals, and there is random matching between primary and secondary labor market participants in forming households (so one quarter of all households are P,P pairs, one quarter S,S pairs, and one half P,S pairs). The tax authority can observe household income but not individual income, and the tax authority does not know the composition of the household.

Household utility is defined as follows:

\[ u = \log[y - t(y)] + N_P \cdot \log[1 - a L_P] + N_S \cdot \log[1 - \beta L_S]. \]

\( y \) denotes labor income, \( t(y) \) taxes as a function of labor income, \( N_P \) the number of primary earners in the household (0, 1 or 2), \( L_P \) the labor supply of primary earners, \( N_S \) the number of secondary earners (0, 1, or 2), and \( L_S \) the labor supply by secondary earners.

(a) Specify the set of household incomes that the tax authority might observe, and over which it must therefore define the tax schedule. Is it possible that all households would appear to be identical from the standpoint of the tax authority?

(b) Consider the following tax schedule, defined as a function of household labor income, \( y \):

\[
\begin{align*}
  t &= 1.5 \quad & y &= 2 \\
  t &= -0.5 \quad & y &< 2.
\end{align*}
\]

\( t > 0 \) means that the household pays taxes; \( t < 0 \) denotes receipt of a subsidy. Verify that this tax schedule will yield no net revenue for the government.