A Discontinuous Galerkin Method for the RANS Equations With ProjectX

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Project X

Research initiative at Aerospace Computational Design Laboratory (at MIT) aimed at developing the next generation CFD capability.

Goals:
- Engineering accuracy in a reasonable amount of time and in an automated manner.
- Allow engineers to confidently use CFD without becoming meshing experts.

Key Features:
- Higher-order discretization using the Discontinuous Galerkin finite element method
- Solution-based adaptivity
- Direct interface to Computer-Aided Design (CAD) models
Compressible Reynolds-Averaged Navier Stokes Equations

Average the NS Equations, using models for complex fluctuation terms:

\[
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\bar{\rho} \tilde{u}_i) = 0
\]  
(1)

\[
\frac{\partial}{\partial t} (\bar{\rho} \tilde{u}_i) + \frac{\partial}{\partial x_j} \left( \bar{\rho} \tilde{u}_j \tilde{u}_i + \bar{p} \delta_{ij} \right) = \frac{\partial}{\partial x_j} \left[ 2 (\mu + \mu_t) \left( \tilde{s}_{ji} - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial x_k} \delta_{ij} \right) \right]
\]  
(2)

\[
\frac{\partial}{\partial t} \left[ \bar{\rho} \left( \tilde{e} + \frac{1}{2} \tilde{u}_i \tilde{u}_i \right) \right] + \frac{\partial}{\partial x_j} \left[ \bar{\rho} \tilde{u}_j \left( \tilde{h} + \frac{1}{2} \tilde{u}_i \tilde{u}_i \right) \right] =
\]  

\[
\frac{\partial}{\partial x_j} \left[ c_p \left( \frac{\mu}{Pr} + \frac{\mu_t}{Pr_t} \right) \frac{\partial \tilde{T}}{\partial x_j} \right] + \frac{\partial}{\partial x_j} \left[ 2 \tilde{u}_i (\mu + \mu_t) \left( \tilde{s}_{ij} - \frac{1}{3} \frac{\partial \tilde{u}_k}{\partial x_k} \delta_{ij} \right) \right]
\]  
(3)

where \( \bar{\cdot} \) and \( \tilde{\cdot} \) denote the Reynolds and Favre averages, \( u_i \) are the components of velocity, and \( s_{ij} \) is the strain rate tensor.

- **Neglect:** turbulent kinetic energy, molecular diffusion, turbulent transport
- **Require:** closure models for the Reynolds stress (Boussinesq Approximation) and eddy viscosity (\( \mu_t \)).
Spalart Allmaras One-Variable Turbulence Model

**SA Overview**
- Closes the RANS Equations with a one-equation model for $\tilde{\nu}$ with $\mu_t = f(\tilde{\nu})$.
- Popular in the aerospace community; closely matches experiment for most attached or mildly separated cases.

**Issues**
- Requires distance to the nearest solid wall.
- Potentially unstable when $\tilde{\nu} < 0$.
- $\tilde{\nu}$ varies rapidly near the BL edge (discontinuous in the 0 viscosity limit).
  - Needs additional mesh resolution at the BL edge as a result.

**Modifications to mitigate $\tilde{\nu} < 0$ issues**
- Soft-limited $\mu_t \geq 0$.
- Modified source so turbulent energy decreases if $\tilde{\nu} < 0$.
- Source production term is non-negative.
Wall-Distance Calculation

Popular Choices

- Solve for distance exactly using a modified level-set equation over the entire flow-field.
- Intelligent-search for the distance at points of interest.

Approach

- Use kd-tree filled with surface quadrature points to perform a nearest-neighbor search from quadrature points in the flow field.
  - Assumes wall-curvature is small relative to quad-point spacing.
- Result initializes a newton search to minimize wall-distance based on a polynomial representation of the geometry.

kd-trees

- kd-trees are essentially k-dimensional binary trees.
- Expected Asymptotic Performance: $O(\log(N)^k)$ to search a tree with $N$ points.
Why DG?

**Discontinuous Galerkin Finite Elements: Advantages**

- Easier development of higher order schemes for convection-dominated flows using upwinding ideas from FD and FV
- Compact stencils (element and nearest neighbors); well-suited to unstructured meshes, mesh adaptation ($h$ and $p$), and parallelization
- No global continuity constraint means maximal geometric flexibility
- $O(h^{p+1})$ solution error; potentially $O(h^{2p})$ output error

**Advantages of Higher Order**

- If the error converges at $O(h^{p+1})$, the computation time is estimated:
  \[ \log T \approx d \left( -\frac{1}{p+1} \log E \right) - \log F + \text{const} \]
  - $d$ is the dimension, $F$ is the computational rate
- For small $E$, the time depends exponentially on $\frac{d}{p+1}$
- Increasing $p$ can significantly lower $T$
DG Discretization

The RANS equations have the strong form:

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{F}^{\text{inv}}(\mathbf{u}) - \nabla \cdot \mathbf{F}^{\text{vis}}(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{S}(\mathbf{u}, \nabla \mathbf{u})
\]  
(4)

The corresponding weak form is obtained (nontrivially!) through multiplication by test functions \( v \) and integration by parts.

- \( \mathbf{F}^{\text{inv}} \): inviscid flux, evaluated using Roe’s method.
- \( \mathbf{F}^{\text{vis}} \): viscous flux, evaluated using the 2nd method of Bassi and Rebay.

Discretization

For some triangulation \((T_h)\) of the domain, apply a polynomial basis \((\mathcal{V}_p)\) locally over each element to the governing equations in weak form.

\[
\sum_{\kappa \in T_h} \int_{\kappa} \mathbf{v}_h^T \frac{\partial \mathbf{u}_h}{\partial t} + R_h(\mathbf{u}_h, \mathbf{v}_h) = 0
\]

where test functions \( \mathbf{v}_h \) and solution \( \mathbf{u}_h \) are represented in the discrete basis \( \mathcal{V}_p^h \) and \( R_h \) is the discrete residual.
Dual Consistency: What is it?

The adjoint (dual) solution is analogous to a Green’s Function relating the source of a PDE (e.g. truncation error) to an output, $J$, computed from the solution of the PDE.

Primal and Dual Problems

- **Discrete Primal**: find $u_h \in \mathcal{V}_h^p$ satisfying $R_h(u_h, v_h) = 0, \forall v_h \in \mathcal{V}_h^p$
- Based on discretizing the continuous PDE $R(u, v) = 0$

- **Discrete Dual**: find $\psi_h \in \mathcal{V}_h^p$ satisfying $R'_h(v_h, \psi_h) = J'_h(v_h), \forall v_h \in \mathcal{V}_h^p$
  and $R'$ denotes a derivative taken wrt components of $u_h$, evaluated at $u_h$
- This is effectively a transpose of the discrete primal.

Dual Consistency

- Loosely, a discretization is **dual consistent** if the exact primal and dual solutions ($u, \psi \in \mathcal{V}$) satisfy the discrete dual problem:
  - $R'_h(v, \psi) = J'_h(v), \forall v \in \{\mathcal{V} + \mathcal{V}_h^p\}$

- Not all primal discretizations lead to dual consistent adjoints!
### Standard Weighting

The direct approach: multiply by the test function and integrate.

\[
R_{h,S,DinC} (w_h, v_h) = - \sum_{\kappa \in T_h} \int_{\kappa} v_h^T S (w_h, \nabla w_h)
\]

Unfortunately, this results in a **dual inconsistent** discretization.

### Modified Standard Weighting

\[
R_{h,S,DC} (w_h, v_h) = - \sum_{\kappa \in T_h} \int_{\kappa} v_h^T S (w_h, \nabla w_h) + \int_{\Gamma_i} [w_h]^T \cdot \{\vec{\beta} (w_h, \nabla w_h, v_h)\} + \int_{\partial \Omega} (u^b - w_h^+) \vec{\beta}_b (w_h, \nabla w_h, v_h) \cdot \vec{n}
\]

Although not obvious, this result is **dual consistent** for the correct choice of “dual” fluxes, \(\vec{\beta}\).
Dual Consistency: Why do we care?

Advantages of Dual Consistent (DC) Discretizations

- Dual inconsistency can lead to sub-optimal primal convergence.
- Adjoint problems important in optimal control, design optimisation, etc.
- Output Error: $|\mathcal{J}(u) - \mathcal{J}(u_h)| \leq O(h^{2p})$ when DC but only $O(h^p)$ when inconsistent.

![Figure](image.png)

Figure: Optimal output error convergence rates obtained for a scalar model problem; notice the difference between DC (left) and standard weighting (right).
RAE 2822 at: $M_\infty = 0.6$, $Re_c = 6.3 \times 10^6$, $\alpha = 2.57^\circ$, using the dual consistent discretization.

Figure: Drag error vs $h$ for $p = 1$ (□), $p = 2$ (○), $p = 3$ (●), $p = 4$ (⊳), and $p = 5$ (×).

Comments

- Error measured against $p = 5$ solution on the finest mesh.
- Optimal rate achieved for $p = 1$ and $p = 2$ but not for higher $p$.
  - $p = 5$ obtains between $O(h^3)$ and $O(h^4)$ due to a non-smooth solution and sub-optimal mesh resolution at the BL edge.
Meshing Issues

- RANS allows drastic reductions in mesh size.
  - Not to mention the elimination of unsteady effects through averaging.
- Ratio between largest and smallest turbulent length scales is $O(Re^{3/4})$.
- Aircraft (typical $Re > 10^7$) in 3D: more than $10^{15}$ elements!
- Solving only for mean values reduces streamwise variation in flow quantities. So to take advantage of this, elements should have aspect ratios of $10^3$ to $10^4$ or more in the BL.
- Forming these grids can be extremely difficult, especially for complex geometries and especially in 3D.
- Poor meshing can result in non-convergence.
  - Many mesh adaptation methods require a steady state solution.
  - Problems is exacerbated in higher-order methods where oscillations are likely in under-resolved regions.
Lack of Robustness

- Robustness issues when $\tilde{\nu} < 0$.
- Our modifications to the SA model do not prevent this; in fact it is still very likely on under-resolved meshes.
- Limiting of $\tilde{\nu}$ is hard because one must respect conservation and all the higher order DOFs.

Solutions Under Investigation in ProjectX

- Apply artificial viscosity in under-resolved regions (e.g. BL edge)
- “Unsteady” adaptation method
- Cut-cells (difficult in 3D due to intersecting higher order polynomial surfaces) alleviate the burden of anisotropic meshing

And there’s always the move to 3D, which will only magnify existing robustness and meshing issues.
Unsteady Adaptation

A robust, automated adaptation procedure for error reduction in steady aerodynamic flow simulations.

Overview: Standard Adjoint Approach
- Refine mesh in regions that maximally affect outputs of interest.
- Output error \( J(u) - J(u_h) \) can be expressed element-wise through the primal residual \( R_h(u, \psi_h - \psi) \) or the dual residual \( R^\psi_h(u_h - u, \psi) \).
  - Approximate exact solutions \( u, \psi \)
  - Average the above error estimates to obtain an element-wise indicator.
  - Adapt mesh based on where the error metric is large.
- Fails when steady state solution cannot be reached.

Overview: Unsteady Approach
- Attempts to improve the robustness of the standard method.
- Add time-dependent term to equations and adapt at each time-step.
- Uses the same adjoint-based error indicator as above.
Figure: Initial Isotropic Mesh (11063 elements): Fine for inviscid flows, but we could not obtain a steady RANS solution.

Figure: Final, Adapted Mesh (11620 elements): Unsteady method produced highly anisotropic elements with substantially more boundary layer resolution.
RANS $p = 3$ Solution: EET Airfoil

The EET 3-element airfoil at: $M_\infty = 0.26$, $Re_c = 9 \times 10^6$, $\alpha = 8^\circ$.

**Mach Number**

$c_l = 3.51$, $c_d = 0.0436$ (error 0.7%) agrees with experiment and BCFD (2nd order FV code at Boeing) results.