Answers to Problem Set Number 03
for 18.311 — MIT (Spring 2008)


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Contents

REGULAR PROBLEMS. 2

1 Haberman’s book, problem 71.01. 2
2 Haberman’s book, problem 71.03. 2
3 Haberman’s book, problem 78.01. 3
4 Haberman’s book, problem 78.03. 4
5 Haberman’s book, problem 79.02. 4

6 Dispersive Waves and Modulations. 5
   6.1 Statement: Dispersive Waves and Modulations . . . . . . . . . . . . . 5
   6.2 Answer: Dispersive Waves and Modulations . . . . . . . . . . . . . . . 6

7 Problem ExPD20: Separation of variables for the 1D wave equation. 8
   7.1 Statement for ExPD20: Separation of variables for the 1D wave equation . . . . 8
   7.2 Answer for ExPD20: Separation of variables for the 1D wave equation . . . . . . . . 9

SPECIAL PROBLEMS. 10

8 Haberman’s book, problem 78.08. 10

9 Problem TFPa17: Shock interaction with a traffic light. 13
   9.1 Statement for TFPa17: Shock interaction with a traffic light . . . . . . 13
   9.2 Answer for TFPa17: Shock interaction with a traffic light . . . . . . . . . 13

List of Figures

3.1 Haberman’s problem 78.01: Characteristics in cases $c_0 < c_1$ and $c_0 > c_1$. . . . . 3
8.1 Haberman’s problem 78.08: Characteristics for a signaling problem. . . . . . . 11
1 Haberman’s book, problem 71.01.

Statement: _______________________________________________________________________
Experiments in the Lincoln Tunnel (combined with theoretical work discussed in exercise 63.7) suggests that the traffic flow is approximately

\[ q(\rho) = a \rho \left( \ln(\rho_{\text{max}}) - \ln(\rho) \right), \]  

\hspace{1cm} (71.01.1)

where \(a\) and \(\rho_{\text{max}}\) are known constants. Suppose that the initial density \(\rho(x, 0)\) varies linearly from bumper-to-bumper traffic (behind \(x = -x_0 < 0\)) to no traffic (ahead of \(x = 0\)), as sketched in figure 71-6. Two hours later, where does \(\rho = \frac{1}{2} \rho_{\text{max}}\)?

Answer: _______________________________________________________________________
The wave velocity \(c = \frac{dq}{d\rho}\), is given by

\[ c = a \left( \ln(\rho_{\text{max}}) - \ln(\rho) - 1 \right). \]  

\hspace{1cm} (71.01.2)

In particular, for \(\rho = \rho_0 = \frac{1}{2} \rho_{\text{max}}\), we have \(c = c_0 = a(\ln 2 - 1)\). Thus, the value \(\rho = \rho_0\) moves in time following the characteristic

\[ x = c_0 t - \frac{1}{2} x_0, \]

since \(\rho \left( -\frac{1}{2} x_0, 0 \right) = \rho_0\) for the given initial data. Substituting \(t = 2\) hours in this last formula gives the desired answer.

2 Haberman’s book, problem 71.03.

Statement: _______________________________________________________________________
Suppose that the flow-density relationship was known only as a specific sketched curve. If the initial traffic density was known, how would you determine where to look to observe the density \(\rho_0\)?

Answer: _______________________________________________________________________
If we know \(q = q(\rho)\) approximately, in order to find out where some density \(\rho_0\) will be observed — for some given initial data — we need to do two things:
(a) Find the position(s) $x = x_0$ where the given density $\rho_0$ occurs for the initial data.

(b) Find/estimate the wave velocity $c_0 = \frac{dq}{d\rho}(\rho_0)$.

Then $\rho_0$ will occur along $x = c_0 t + x_0$.

3 Haberman’s book, problem 78.01.

**Statement:**

Suppose that the initial traffic density is

$$\rho(x, 0) = \begin{cases} 
\rho_0 & \text{for } x < 0, \\
\rho_1 & \text{for } x > 0,
\end{cases}$$

(78.01.1)

where $\rho_0$ and $\rho_1$ are constants. Consider the two cases, $\rho_0 < \rho_1$, and $\rho_0 > \rho_1$. For which of the preceding cases is a density shock necessary? Briefly explain.

**Answer:**

Figure 3.1: Haberman’s problem 78.01. Left: characteristics for the case $\rho_0 > \rho_1$. Then $c_0 < c_1$, the characteristics diverge, and an expansion fan is needed. Right: characteristics for the case $\rho_0 < \rho_1$. That $c_0 > c_1$, the characteristics converge, and a shock is needed.
(a) Case $\rho_0 < \rho_1$.
For traffic flow, the characteristic (or density wave) speed $c$ is a decreasing function of the density. Thus, in this case the characteristics starting (at $t = 0$) on $x < 0$, will have a value of $c$ which is larger than the value for the characteristics starting on $x > 0$. That is: $c_0 > c_1$. It follows that the characteristics staring on $x < 0$ will cross those starting on $x > 0$, so that a shock will be needed. See the right frame in figure 3.1.

(b) Case $\rho_0 > \rho_1$.
The same argument used in part (a) shows that in this case the characteristics starting on $x < 0$ will not cross those starting on $x > 0$. In fact, they will leave a gap in space–time. Thus, in this case no shock arises. In fact, this case gives rise to an expansion fan. See the left frame in figure 3.1.

4 Haberman’s book, problem 78.03.

Statement: Assume that $u = u_{\max} (1 - \rho/\rho_{\max})$ and at $t = 0$ the traffic density is

$$\rho(x, 0) = \begin{cases} 
(1/3) \rho_{\max} & \text{for } x < 0, \\
(2/3) \rho_{\max} & \text{for } x > 0.
\end{cases}$$  \tag{78.03.1}

Why does the density not change in time?

Answer: In this case the car flow is given by $q = \rho u = u_{\max} (1 - \rho/\rho_{\max}) \rho$ and the density wave velocity by $c = \frac{dq}{d\rho} = u_{\max} (1 - 2 \rho/\rho_{\max})$. Since $c(\rho_{\max}/3) > c(2 \rho_{\max}/3)$, the initial conditions in (78.03.1) above give rise to a shock. Since $q(\rho_{\max}/3) = q(2 \rho_{\max}/3)$, this shock propagates at zero velocity. It follows that the solution (for all times) of the problem with the initial conditions above in (78.03.1) is:

$$\rho(x, t) = \begin{cases} 
(1/3) \rho_{\max} & \text{for } x < 0 , \\
(2/3) \rho_{\max} & \text{for } x > 0.
\end{cases}$$  \tag{78.03.2}

5 Haberman’s book, problem 79.02.

Statement: Suppose that

$$\rho(x, 0) = \begin{cases} 
\rho_0 & \text{for } x > 0, \\
0 & \text{for } x < 0.
\end{cases}$$  \tag{79.02.1}
Determine the velocity of the shock. Briefly give a physical explanation of the result.

**Answer:**

Since $c$ is a decreasing function of $\rho$, $c(0) > c(\rho_0)$ and these initial conditions give rise to a shock, with speed:

$$s = \frac{q(\rho_0) - q(0)}{\rho_0 - 0} = \frac{q(\rho_0)}{\rho_0} = u(\rho_0).$$  \hspace{1cm} (79.02.2)

This result should not be surprising, since the shock here is just the position of the last car in a uniform stream of traffic at density $\rho_0$. Obviously, this car moves at speed $u(\rho_0)$.

### 6 Dispersive Waves and Modulations.

#### 6.1 Statement: Dispersive Waves and Modulations.

Consider the following linear partial differential equations for the scalar function $u = u(x, t)$:

$$u_t + cu_x + du_{xxx} = 0, \quad (6.1)$$

$$u_{tt} - u_{xx} + au = 0, \quad (6.2)$$

$$iu_t + bu + gu_{xx} = 0, \quad (6.3)$$

where $c, d, a, b,$ and $g$ are constants, and we will assume that the equations have been nondimensionalized.\(^1\) It should be clear that, in all three cases,

$$u = Ae^{i(kx - \omega t)}, \quad \text{where } \omega = \Omega(k), \quad (6.4)$$

is a solution of the equations, for any constants $A$ and $k$, provided that we take

- **M1.** For equation (6.1): $\Omega(k) = ck - dk^3$  \hspace{1cm} **Verify that this is true.**

- **M2.** For equation (6.2): $\Omega(k) = \sqrt{a + k^2}$  \hspace{1cm} **Verify that this is true.**

- **M3.** For equation (6.3): $\Omega(k) = -b + gk^2$  \hspace{1cm} **Verify that this is true.**

Solutions such as that in (6.4) represent monochromatic traveling waves, with amplitude $|A|$, wave number $k$, and angular frequency $\omega$. **Note that in all these cases, $\Omega$ is NOT a linear function of $k$.** Thus we say that the equations are dispersive and call $\Omega$ the dispersion function.

**Remark 6.1** For a dispersive system waves with different wavelengths propagate at different speeds. Thus a localized wave packet, made up of many waves of different wavelengths, will disperse in time — as the waves cease to add up in the proper phases to guarantee a localized wave-packet.

---

\(^1\)These equations arise in many applications, but we will not be concerned with these applications here.
Your task: consider a dispersive waves system, that is: a system of equations accepting monochromatic traveling waves as solutions, provided that their wave number \( k \) and angular frequency \( \omega \) are related by a dispersion relation

\[
\omega = \Omega(k) .
\] (6.5)

Consider now a slowly varying, nearly monochromatic solution of the system. To be more precise: consider a solution such that at each point in space–time one can associate a local wave number \( k = k(x, t) \) and a local angular frequency \( \omega = \omega(x, t) \). In particular, both \( k \) and \( \omega \) vary slowly in space and time, so that they change very little over a few wavelengths or a few wave periods. Note though that they may change considerably over many wavelengths or wave periods! Then

Assuming conservation of wave crests, derive equations governing \( k \) and \( \omega \).

These equations are called the **Wave Modulation Equations**.

Remark 6.2 Notice that the assumption that \( k \) and \( \omega \) vary slowly is fundamental in making sense of the notion of a locally monochromatic wave. To even define a wave number or an angular frequency, the wave must look approximately monochromatic over several wavelengths and periods.

Remark 6.3 Why is it reasonable to assume that the wave crests are conserved? The idea behind this is that, for a wave crest to disappear (or for a new wave crest to appear), something pretty drastic has to happen in the wave field. This is not compatible with the assumption of slow variation. It does not mean that it cannot happen, just that it will happen in circumstances where the assumption of slow variation is invalid. There are some pretty interesting open research problems in pattern formation that are related to this point.

Hint 6.1 It should be clear that one of the equations is \( \omega = \Omega(k) \), since the solution behaves locally like a monochromatic wave. For the second equation, express the density of wave crests (and its flux) in terms of \( k \) and \( \omega \). Then write the equation for the conservation of wave crests using these quantities.

6.2 Answer: Dispersive Waves and Modulations.

The wavelength is related to the wave number by \( \lambda = \lambda(x, t) = \frac{2\pi}{k} \), while for the wave period \( \tau \) we have \( \tau = \tau(x, t) = \frac{2\pi}{\omega} \). Thus:

\[ \frac{1}{\lambda} = \frac{1}{2\pi} k \] is the number of waves per unit length. That is: the **wave density**.
M5. \( \frac{1}{\tau} = \frac{1}{2\pi} \omega \) is the number of waves per unit time. That is: the wave flux.

Hence, if the waves are conserved, we can write the conservation of wave crests equation:\(^2\)

\[ k_t + \omega_x = 0, \]  
(6.6)

where \( \omega = \Omega(k) \), and we have eliminated the common \( 2\pi \) factor.

**Remark 6.4** Note that the crests in the wave profile move at the speed

\[ c_p = c_p(k) = \frac{\omega}{k} = \frac{\Omega(k)}{k}, \]

as it is easy to see by noticing that we can write \( \exp\{i (k x - \omega t)\} = \exp\{i k (x - c_p t)\} \) in (6.4).

Since \( \omega = c_p k \), this velocity plays in this theory the same role as the car speed in Traffic Flow. It is called the phase speed.

**Remark 6.5** Substituting \( \omega = \Omega(k) \) into equation (6.6), and using the chain rule, we obtain:

\[ k_t + c_g(k) k_x = 0, \]  
(6.7)

where \( c_g = \frac{d\Omega}{dk} \) is called the group speed, and it plays the same role as the characteristic wave speed in Traffic Flow. Changes in \( k \) and \( \omega \) travel at this speed. In particular, a “wave package” will travel at this speed — and it can be shown that it is also the speed at which the wave energy propagates.

**Remark 6.6** You may have encountered the notion of group speed in earlier courses, when studying the beating phenomena. Namely, adding two sinusoidal waves of close frequency and wavelength:

\[ \Phi = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t), \]  
(6.8)

where \( k_1 \approx k_2 \) and \( \omega_1 \approx \omega_2 \), produces “beats” propagating at a velocity:

\[ \frac{\Delta \omega}{\Delta k} \approx c_g = \text{group speed}. \]

This follows because we can write

\[ \Phi = 2 \cos \left( \frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t \right) \cos \left( \frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t \right) \]
\[ = 2 \cos \left( \frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right) \cos \left( \bar{k} x - \bar{\omega} t \right), \]  
(6.9)

where \( \bar{k} \) and \( \bar{\omega} \) are the average wave number and angular frequency. The expression above is the simplest example of a slowly modulated wave, with the modulation provided by the first factor in the second formula for \( \Phi \).

\(^2\)Note the similarity with the Traffic Flow conservation of cars equation.
Remark 6.7 Where did we use the fact that $k$ and $\omega$ vary “slowly” in space and time?

This gets used twice:

- **First**: So that one can talk about a wavelength and a frequency for the waves. For these things to make sense the wave must (locally) look like a plane wave.

- **Second**: when we write $\omega = \Omega(k)$ for the variables $k$ and $\omega$. This is the relationship that applies to plane, monochromatic waves — where $k$ and $\omega$ are constants. Thus, to be (approximately) valid in the variable case, the changes in $k$ and $\omega$ must be small over a few wave periods or wavelengths.

7 Problem ExPD20:

Separation of variables for the 1D wave equation.

7.1 Statement for the problem ExPD20:

Separation of variables for the 1D wave equation.

Consider the wave equation

$$u_{tt} - u_{xx} = 0.$$  \hfill (7.1)

Elsewhere it was shown that any solution of this equation can be written in the form:

$$u = f(x - t) + g(x + t),$$  \hfill (7.2)

where $f$ and $g$ are arbitrary functions. **Find ALL the separation of variables solutions to the wave equation (7.1), for the cases below** — the separation of variables solutions are those that have the form $u = T(t)X(x)$, for some functions $T = T(t)$ and $X = X(x)$.

- **Case A.** The solution $u = u(x, t) = T(t)X(x)$ is periodic in $x$, of period $2\pi$.

- **Case B.** The solution $u = u(x, t) = T(t)X(x)$ vanishes at $x = 0$, and at $x = \pi$.

This second case corresponds to the problem of a string of (nondimensional) length $\pi$, tied at both ends. The solutions that you will find are the *standing wave* modes for the wave equation.

In each case, show that the solutions you have found have the form given in equation (7.2) above. In particular, this will show you how the standing wave solutions can be written as a combination of a left and a right moving waves.
7.2 Answer for the problem ExPD20:

Separation of variables for the 1D wave equation.

Substituting the form $u = X(x)T(t)$ into the wave equation, and dividing by $u = X(x)T(t)$, we obtain:

$$\frac{1}{T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2}.$$ 

Since the left hand side on this equation is a function of time only, and the right hand side is a function of space only, both must be constants. Thus we obtain:

$$\frac{1}{T} \frac{d^2T}{dt^2} = \mu, \quad \text{and} \quad \frac{1}{X} \frac{d^2X}{dx^2} = \mu, \quad (7.3)$$

where $\mu$ is some constant. Then we find:

- **Case A.** Since $X$ is periodic, of period $2\pi$, we must have:

  $$X(x) = a \cos(nx + x_0), \quad (7.4)$$

  where $\mu = -n^2$, $n$ is an integer, and $a$ and $x_0$ are constants. Then

  $$T(t) = b \cos(nt + t_0), \quad (7.5)$$

  where $b$ and $t_0$ are constants. Thus

  $$T(x) = ab \cos(nx + x_0) \cos(nt + t_0) \quad (7.6)$$

  $$= \frac{ab}{2} \cos(n(x + t) + x_0 + t_0) + \frac{ab}{2} \cos(n(x - t) + x_0 - t_0), \quad (7.7)$$

  which, clearly, has the form in (7.2).

- **Case B.** Since $X$ must vanish both at $x = 0$ and at $x = \pi$, we must have:

  $$X(x) = a \sin(nx), \quad (7.8)$$

  where $\mu = -n^2$, $n$ is an integer, and $a$ is a constant. Then

  $$T(t) = b \cos(nt + t_0), \quad (7.9)$$

  where $b$ and $t_0$ are constants. Thus

  $$T(x) = ab \sin(nx) \cos(nt + t_0) \quad (7.10)$$

  $$= \frac{ab}{2} \sin(n(x + t) + t_0) + \frac{ab}{2} \sin(n(x - t) - t_0), \quad (7.11)$$

  which, clearly, has the form in (7.2).
SPECIAL PROBLEMS.

8 Haberman’s book, problem 78.08.

Statement: Determine the traffic density on a semi–infinity \((x > 0)\) highway for which the density at the entrance is

\[
\rho(0, t) = \begin{cases} 
\rho_1 & \text{for } 0 < t < \tau, \\
\rho_0 & \text{for } \tau < t,
\end{cases} \quad (78.08.1)
\]

where \(\tau > 0\) is constant, and the initial density is uniform along the highway — assume that \(\rho(x, 0) = \rho_0\), for \(x > 0\). Furthermore, assume that \(\rho_1\) is lighter traffic than \(\rho_0\), and that both are light traffic; in fact assume that \(u(\rho) = u_{\text{max}} \left(1 - \rho/\rho_{\text{max}}\right)\) and that \(\rho_1 < \rho_0 < \rho_{\text{max}}/2\). Sketch the density at various values of time.

Answer: We have \(q = \rho u(\rho) = u_{\text{max}} (\rho - \rho^2/\rho_{\text{max}})\), so that the traffic flow is a quadratic function of \(\rho\), with the wave speed \(c = u_{\text{max}} \left(1 - 2 \rho/\rho_{\text{max}}\right)\) linear in \(\rho\). In this case \(c\) itself is conserved, and it turns out to be more convenient to solve the problem directly in terms of \(c\) — and recover \(\rho\) later from

\[
\rho = \frac{1}{2} \rho_{\text{max}} \left(1 - \frac{c}{u_{\text{max}}}\right). \quad (78.08.2)
\]

The problem for \(c\) is

\[
c_t + \left(\frac{1}{2} c^2\right)_x = 0, \quad (78.08.3)
\]

with

\[
c(0, t) = \begin{cases} 
c_1 & \text{for } 0 < t < \tau, \\
c_0 & \text{for } \tau < t,
\end{cases} \quad \text{and} \quad c(x, 0) = c_0 \quad \text{for } x > 0, \quad (78.08.4)
\]

where \(c_1 = c(\rho_1) > c(\rho_0) = c_0\). For this problem the shock speed is the average of \(c\) on each side of the shock, and the characteristics are straight lines in space–time, along which \(c\) is constant, with slope \(c\). We solve it next, starting by computing the characteristics (see figure 8.1), and then placing shocks whenever needed.

(a) Characteristics starting on \(x > 0\), at \(t = 0\).
These are given by \(x = c_0 t + \xi\), with \(\xi > 0\), and \(c = c_0\) on them.

(b) Characteristics starting on \(0 < t < \tau\), at \(x = 0\).
These are given by \(x = c_1 (t - \xi)\), with \(0 < \xi < \tau\), and \(c = c_1\) on them.

(c) Characteristics starting on \(t = \tau\), at \(x = 0\).
These are given by \(x = c (t - \tau)\), with \(c_0 \leq c \leq c_1\), and \(c\) constant on each of them (expansion fan arising because \(c_1 > c_0\)).
(d) Characteristics starting on $\tau < t$, at $x = 0$.

These are given by $x = c_0 (t - \xi)$, with $\tau < \xi$, and $c = c_0$ on them.

We now have the following situation:

A. Because $c_1 > c_0$, the characteristics in (a) and (b) above cross. Thus a shock separating the constants states $c_0$ on the right, from $c_1$ on the left arises. The shock path and speed are
given by (at least initially)
\[ x = c_s \, t \quad \text{for} \quad 0 \leq t \leq t_c, \quad \text{where} \quad c_s = \frac{1}{2} (c_0 + c_1), \quad (78.08.5) \]
and \( t_c \) is the time at which the first characteristic in the expansion fan reaches the shock (this happens at \( x = x_c \)). Namely:
\[ x_c = c_s \, t_c = c_1 (t_c - \tau) \quad \Longrightarrow \quad t_c = \frac{2 \, c_1 \, \tau}{c_1 - c_0} \quad \text{and} \quad x_c = \frac{c_1 + c_0}{c_1 - c_0} \, c_1 \, \tau. \quad (78.08.6) \]
This corresponds to the characteristic starting at the initial data (as in (a) above) for which \( x_c = c_0 \, t_c + \xi_c \), which yields
\[ \xi_c = x_c - c_0 \, t_c = c_1 \, \tau. \quad (78.08.7) \]
This characteristic is also drawn in figure 8.1.

B. In the expansion fan region the solution (as follows from (c)) is given by
\[ c = \frac{x}{t - \tau}. \quad (78.08.8) \]

C. After \( t = t_c \) the shock no longer connects the constant state \( c_1 \) on the left, with the constant state \( c_0 \) on the right. Rather, it connects the expansion fan on the left with the constant state \( c_0 \) on the right — see figure 8.1. Thus, for \( t > t_c \), the shock position follows from the solution of the following O.D.E.:
\[ \frac{dx_s}{dt} = \frac{1}{2} \left( c_0 + \frac{x_s}{t - \tau} \right), \quad (78.08.9) \]
where \( x = x_s(t) \) denotes the shock position, \( x_s(t_c) = x_c \), and we have used equation (78.08.8) in evaluating the state on the left of the shock. This has the general solution
\[ x_s = c_0 \, (t - \tau) + C \, \sqrt{t - \tau}, \quad (78.08.10) \]
where the constant \( C \) follows from the condition \( x_s(t_c) = x_c \). Using (78.08.6) it then follows that \( C = \sqrt{(c_1^2 - c_0^2) \, \tau} \).

Notice that, \( x_s(t) > c_0 \, (t - \tau) \) for all \( t > t_c \). Thus, the state on the left of the shock, for \( t > t_c \), is always the expansion fan (since the left edge of the expansion fan, given by the characteristic \( x = c_0 \, (t - \tau) \), never reaches the shock). In other words: equation (78.08.10) applies for all times \( t \geq t_c \).

We can now put it all together and explicitly write the solution to the problem in (78.08.1) as follows (see figure 8.2):
\[ c(x, t) = \begin{cases} 
  c_1 & \text{for} \quad 0 \leq x < c_s \, t, \\
  c_0 & \text{for} \quad c_s \, t < x,
\end{cases} \quad \text{for} \quad 0 \leq t \leq \tau, \quad (78.08.11) \]
\[
c(x, t) = \begin{cases} 
c_0 \frac{x}{t - \tau} & \text{for } 0 \leq x < c_0 (t - \tau), \\
c_1 \frac{x}{t - \tau} & \text{for } c_0 (t - \tau) \leq x < c_1 (t - \tau), \\
c_0 & \text{for } c_1 (t - \tau) \leq x < c_s t, \\
c_0 & \text{for } c_s t < x,
\end{cases}
\text{ for } \tau < t < t_c, \quad (78.08.12)
\]

and
\[
c(x, t) = \begin{cases} 
c_0 \frac{x}{t - \tau} & \text{for } 0 \leq x < c_0 (t - \tau), \\
c_0 & \text{for } c_0 (t - \tau) \leq x < x_s(t), \\
c_0 & \text{for } x_s(t) < x,
\end{cases}
\text{ for } \tau < t < t_c, \quad (78.08.13)
\]

where
\[
c_s = \frac{1}{2} (c_0 + c_1), \quad x_s = c_0 (t - \tau) + \sqrt{(c_1^2 - c_0^2) \tau (t - \tau)},
\]

and \(\rho\) follows from \(c\) using equation (78.08.2). Finally, notice that the value right behind the shock is given by
\[
c = \frac{x_s}{t - \tau} = c_0 + \sqrt{\frac{(c_1^2 - c_0^2) \tau}{t - \tau}}. \quad (78.08.14)
\]


9.1 Statement for TFPa17: Shock interaction with a traffic light.
At time \(t = 0\), the traffic pattern on a long highway consists of two sections of constant concentration, joined by a shock which moves in the positive \(x\) direction, as shown in figure 9.1. If a traffic light at \(x = 0\) turns (at time \(t = 0\)) and remains red, describe the resultant motion. Let the position of the shock at time \(t = 0\) be given by \(x = -L < 0\), and assume that \[q = \frac{4 q_m}{\rho_j^2} \rho (\rho_j - \rho)\] — with \(0 < \rho_0 < \rho_1 < (1/2) \rho_j = \rho_m\).

9.2 Answer for TFPa17: Shock interaction with a traffic light.
In this case the characteristic speed is given by
\[
c = \frac{dq}{d\rho} = \frac{4 q_m}{\rho_j^2} (\rho_j - 2 \rho). \quad \text{Furthermore: } c_0 = c(\rho_0) > c_1 = c(\rho_1) > 0. \quad (9.1)
\]
Because the traffic flow is a quadratic function of \(\rho\), it follows that: For any shock, the shock speed is the average of the characteristic speeds on each side of the shock. In particular, for the shock in figure 9.1
\[
c_0 > U = \frac{1}{2} (c_0 + c_1) > c_1 > 0. \quad (9.2)
\]
Figure 8.2: Haberman’s problem 78.08. Solution for the initial-boundary value problem given in equation (78.08.1), as described by equations (78.08.11–78.08.13). Top-left: solution for a typical time $0 < t \leq \tau$. Top-right: solution for a typical time $\tau < t < t_c$. Bottom: solution for a typical time $t_c < t$ (as $t \to \infty$ the jump at the shock vanishes like $1/\sqrt{t - \tau}$).

Note also that

$$c_M = c(0) = \frac{4 \rho_0}{\rho_j} = c(\rho_j) > c_0.$$ \hspace{1cm} (9.3)

We now solve the problem, beginning with the computation of all the characteristics.

**c1. Characteristics starting, at $t = 0$, on $x = s < -L$.**

These carry the value $\rho = \rho_0$, and are given by $x = c_0 t + s$.
They cover the region $x < c_0 t - L$ for $t > 0$.

**c2. Characteristics starting, at $t = 0$, on $-L < x = s < 0$.**

These carry the value $\rho = \rho_1$, and are given by $x = c_1 t + s$.
They cover the region $c_1 t - L < x < c_1 t$ for $t > 0$.

**c3. Characteristics starting, at $t = \tau > 0$, on the left side of the t-axis $x = 0$.**

These carry the value $\rho = \rho_j$, and are given by $x = -c_M (t - \tau)$.
They cover the region $-c_M t < x < 0$ for $t > 0$. 

\[ \rho = \rho(x, t) \text{ for } 0 < t \leq \tau \]

\[ \rho = \rho(x, t) \text{ for } \tau < t < t_c \]
c4. **Characteristics starting, at** \( t = \tau > 0 \), **on the right side of the t-axis** \( x = 0 \).

These carry the value \( \rho = 0 \), and are given by \( x = c_M (t - \tau) \).

They cover the **region** \( 0 < x < c_M t \) for \( t > 0 \).

c5. **Characteristics starting, at** \( t = 0 \), **on** \( 0 < x = s \).

These carry the value \( \rho = \rho_1 \), and are given by \( x = c_1 t + s \).

They cover the **region** \( c_1 t < x \) for \( t > 0 \).

Since \( c_M > c_0 > c_1 > 0 \), there are plenty of crossings of characteristics, so shocks are needed.

**DRAW THE FIVE REGIONS DEFINED ABOVE, AND SEE HOW THEY OVERLAP.**

It is easy to see that the shocks needed to resolve the crossings of the characteristics, as defined above, are:

s1. The shock depicted in figure 9.1 continues moving to the right, at velocity \( U \), along the path \( x = U t - L \), till it meets the shock described in item s2.
s2. A shock starts at the light, and moves backward separating the cars still approaching the light (at density $\rho_1$), from the cars already stopped behind the light (at density $\rho_j$). The path of this shock is given by

$$x = U_s t,$$

where $U_s = (c_1 - c_M)/2 < 0$.

This shock meets with the one described in item s1 at

$$x = x_1 = \frac{U_s L}{U - U_s} \quad \text{and} \quad t = t_1 = \frac{L}{U - U_s}.$$

When the shocks described in items s1 and s2 meet, they merge into the single shock described in item s3.

s3. For $t > t_1$, there is a shock along the path $x = x_1 + U_M (t - t_1)$, separating the cars still approaching the light (at density $\rho_0$), from the cars already stopped behind the light (at density $\rho_j$). This shock velocity is

$$U_M = (c_0 - c_M)/2 < 0.$$

s4. A shock starts at the light, and moves forward, separating the cars that went through the light (density ahead of shock $\rho = \rho_1$) from the empty road behind ($\rho = 0$). This shock corresponds to the last car through the light, and has the path

$$x = U_L t,$$

where $U_L = (c_1 + c_M)/2 > 0$.

Note that $U_L$ is equal to the car velocity corresponding to the density $\rho_1$.

The full solution to the problem is then given by:

- $\rho = \rho_0$ for $x < U t - L$ and $0 \leq t \leq t_1$.
- $\rho = \rho_0$ for $x < x_1 + U_M (t - t_1)$ and $t_1 \leq t$.
- $\rho = \rho_1$ for $U t - L < x < U_s t$ and $0 \leq t < t_1$.
- $\rho = \rho_j$ for $U_s t < x < 0$ and $0 < t \leq t_1$.
- $\rho = \rho_j$ for $x_1 + U_M (t - t_1) < x < 0$ and $t_1 \leq t$.
- $\rho = 0$ for $0 < x < U_L t$ and $0 < t$.
- $\rho = \rho_1$ for $U_L t < x$ and $0 \leq t$.

THE END.