Discontinuous Galerkin solution of the Reynolds-averaged Navier–Stokes and $k$–$\omega$ turbulence model equations

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Abstract

Discontinuous Galerkin methods, originally developed in the advective case, have been successively extended to advection–diffusion problems, and are now used in very diverse applications. We here consider the numerical solution of the compressible Reynolds-averaged Navier–Stokes and $k$–$\omega$ turbulence model equations by means of DG space discretization and implicit time integration. Detailed description of the DG discretization of the viscous part of the equations and of several implementation details of the $k$–$\omega$ turbulence model are given. To assess the performance of the proposed methodology we present the results obtained in the computation of the turbulent flow over a flat plate and of the turbulent unsteady wake developing behind a turbine blade.

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1. Introduction

Discontinuous Galerkin (DG) methods are finite element methods in which the solution is approximated by means of piecewise continuous functions inside elements (polynomials in most cases) with no global continuity requirement. The numerical solution is therefore in general discontinuous at element interfaces. The lack of global continuity constraints lead to discrete
approximations characterized by the greatest geometrical flexibility, which is the main reason why this class of methods raised the interest of the scientific community and are currently finding use in very diverse applications. High-order accurate schemes can in fact be constructed with no conceptual difficulty for both structured and unstructured grids. The enforcement of high-order accurate and general boundary conditions is straightforward. Non-conforming grids (i.e. grids with “pending” or “hanging” nodes in the FV jargon), can be treated with no difficulty as well. Elements of different order of accuracy can be easily accommodated in the same grid thus opening the way to a straightforward implementation of “hp” adaptive solution strategies. In addition, DG methods lead to very compact space discretization formulae. The compactness of the scheme is particularly advantageous when an implicit time advancement scheme is employed and/or for a parallel implementation of the method.

For purely advective terms such as those occurring in the inviscid Navier–Stokes (Euler) equations, the jumps are treated by means of (approximate) Riemann solvers. In this respect DG methods are therefore analogous to cell centered upwind finite volume (FV) methods, and, in fact, reduce to the latter in the case of piecewise constant approximations. Higher-order accuracy is however achieved in DG methods by means of higher-order polynomials inside the elements, and in this respect these schemes are therefore similar to the classical (continuous) finite element method. The DG treatment of diffusive terms such as the viscous terms of the Navier–Stokes equations is instead quite different from that commonly employed in FV methods. In fact, in FV schemes the solution is usually considered discontinuous at the cell interfaces in the discretization of the advective terms but it is instead regarded as globally continuous in the approximation of the diffusive terms. The DG discretization of both advection and diffusion operators is instead directly constructed on the same space of discontinuous functions.

Several DG methods for the discretization of advection–diffusion or purely elliptic problems have been proposed in the literature and one of the objectives of this paper is to illustrate one of them, the so-called BR2 method originally introduced by the authors [1] and subsequently theoretically analyzed (and named “BR2” scheme) in [2,3]. The other objective of the paper is to present the application of DG space discretization and Runge–Kutta implicit time integration in the solution of the Reynolds-averaged compressible Navier–Stokes (RANS) equations coupled with the high- or low- Reynolds number k–ω turbulence model, see e.g. [4]. It is well known that the numerical integration of the RANS equations coupled with a two-equations turbulence models such as the k–ω model here considered is a challenging task because of the extreme stiffness of the problem. In practice it is mandatory to supplement the RANS and turbulence model equations with some form of limiting of the magnitude of the computed turbulent quantities in order to prevent the blow-up of the simulations. This turned out to be a particularly important issue for DG approximations, especially for the third- and fourth-order accurate computations at the large CFL numbers allowed by the implicit time integration scheme. In practice, a non-standard implementation of the k–ω model, whereby the logarithm of omega rather than omega itself is used as unknown, has been found very useful to enhance the stability of the method, see e.g. [5]. A further significant improvement in the robustness and efficiency of the method have been obtained by enforcing realizability constraints on the turbulence model. To assess the performance of the proposed methodology we present the results obtained in the computation of the turbulent flow along a flat plate and in the computation of the turbulent unsteady wake behind a turbine blade.
The outline of the paper is as follows. In Section 2 we present the RANS and $k-\omega$ turbulence model equations and we illustrate at length the implementation details required by the turbulence model. Section 3 is devoted to the illustration of the DG space discretization method. Particular emphasis is given to the treatment of the viscous terms. Section 4 describes the issues related to implicit time integration. Section 5 shows the numerical results obtained, and Section 6 contains the conclusions.

### 2. Governing equations

The complete set of the RANS and $k-\omega$ equations can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \quad (1)$$

$$\frac{\partial}{\partial t}(\rho e_0) + \frac{\partial}{\partial x_j}(\rho u_j h_0) = \frac{\partial}{\partial x_j} \left[ u_i \tilde{\tau}_{ij} - q_j \right] - \tau_{ij} \frac{\partial u_i}{\partial x_j} + \beta \rho \bar{\kappa} \bar{e}^{\bar{\omega}}, \quad (2)$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = - \frac{\partial p}{\partial x_j} + \frac{\partial \tilde{\tau}_{ij}}{\partial x_j}, \quad (3)$$

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho u_j k) = \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta \rho \bar{\kappa} \bar{e}^{\bar{\omega}} + \frac{\partial}{\partial x_j} \left[ (\mu + \sigma \bar{\mu}_i) \frac{\partial k}{\partial x_j} \right], \quad (4)$$

$$\frac{\partial}{\partial t}(\rho \bar{\omega}) + \frac{\partial}{\partial x_j}(\rho u_j \bar{\omega}) = \frac{\alpha}{k} \tau_{ij} \frac{\partial u_i}{\partial x_j} - \beta \rho \bar{e}^{\bar{\omega}} + \left( \mu + \sigma \bar{\mu}_i \right) \frac{\partial \bar{\omega}}{\partial x_k} \frac{\partial \bar{\omega}}{\partial x_k} + \frac{\partial}{\partial x_j} \left[ (\mu + \sigma \bar{\mu}_i) \frac{\partial \bar{\omega}}{\partial x_j} \right], \quad (5)$$

$$p = (\gamma - 1) \rho (e_0 - u_k u_k / 2), \quad (6)$$

$$q_j = - \left( \frac{\mu}{Pr} + \frac{\bar{\mu}_i}{Pr_t} \right) \frac{\partial h}{\partial x_j}, \quad (7)$$

$$\tau_{ij} = 2 \bar{\mu}_i \left[ S_{ij} - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] - \frac{2}{3} \rho \bar{k} \delta_{ij}, \quad (8)$$

$$\tilde{\tau}_{ij} = 2 \mu \left[ S_{ij} - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] + \tau_{ij}, \quad (9)$$

$$\bar{k} = \max (0, k) \quad \bar{\mu}_i = \sigma \rho \bar{\kappa} \bar{e}^{-\bar{\omega}}, \quad (10)$$

where $\gamma$, $Pr$ and $Pr_t$ are the ratio of gas specific heats, the molecular and turbulent Prandtl numbers (constant for perfect gases) and

$$S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

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is the mean strain-rate tensor. The value of the $k$–$\omega$ closure parameters $\alpha, \alpha', \beta, \beta', \sigma, \sigma'$ are those of the low-Reynolds $k$–$\omega$ model, see [4].

2.1. The $k$–$\omega$ turbulence model equations

Some remarks are useful to clarify the form of Eqs. (2), (4) and (5). First of all we notice that in Eq. (5) the new variable $\tilde{\omega} = \ln \omega$ appears instead of $\omega$. The use of the logarithm of turbulence variables has been introduced by Ilinca and Pelletier in [6] in the context of the $k$–$\varepsilon$ model using wall functions and it is attractive because the positivity of the turbulence variables is guaranteed. Moreover, the distribution of the logarithmic turbulence variables is much smoother than that of the turbulence variables themselves. However, applying the idea of logarithmic variables to the $k$–$\omega$ model, which is integrated down to the wall without using wall functions, we have found useful to introduce the logarithm of $\omega$ but not the logarithm of $k$. In fact, the near-wall distribution of $\ln \omega$ is much better behaved than that of $\omega$, whilst the solid wall boundary condition for $k$ ($k = 0$) leads to an infinite value of $\ln k$. To deal with possible negative values of $k$ we have used the limited value $\tilde{k}$ and the related eddy viscosity $\bar{\mu}_t$, given by Eq. (10), in the source terms of Eqs. (2), (4) and (5). Notice that $\tilde{k}$ has been limited exactly to zero (and not to an arbitrary small value) because, after the appropriate substitutions, no term in the above equations is divided by $\tilde{k}$. The important thing to observe, however, is that the solution of Eq. (4) is not limited at all and, therefore, $k$ could take also negative values. Using the outlined approach for the solution of the turbulence variables, we have noticed, in practice, that negative values of $k$ may occur during the time evolution of the solution or may even be present in a converged steady state solution. However, we have also found that the occurrence of negative $k$ values can be eliminated by refining the computational grid and/or by increasing the degree of the polynomial approximation. As a final comment on the equations, notice that a source term is present in the mean-flow energy equation because the total energy $e_0$ and the total enthalpy $h_0$ do not include the turbulent kinetic energy, see e.g. [7].

Besides the limited value $\tilde{k}$, we also notice that in the source terms of Eqs. (2), (4) and (5) and in the eddy viscosity defined by Eq. (10) the variable $\tilde{\omega}_r$ is employed instead of $\tilde{\omega}$. This is to indicate that $\tilde{\omega}_r$ fulfills the “realizability” conditions outlined in the following.

2.2. Realizability conditions

Even if the variable $\tilde{\omega}$ precludes the attainment of negative $\omega$ values and significantly improves the stability and accuracy of the computed turbulent quantities, we have observed that sometimes very small values of $\tilde{\omega}$ can induce sudden breakdown of the computations. This can mostly happen when approaching the steady state solution through an unphysical transient which, in an implicit method, can be particularly violent.

In nearly all cases the breakdown can be avoided by reducing the time step magnitude; however, in this way we pay for the computational cost of an implicit method without exploiting its advantages. A better solution for this problem can be devised by enforcing the respect of the “realizability conditions” for the turbulent stresses. This means assuring that the turbulence model predicts positive normal turbulent stresses and satisfies the Schwarz inequality for the shear
turbulent stresses. In the case of the $k–\omega$ model both requirements can be satisfied by imposing a lower bound for $\bar{\omega}$. In fact, the realizability conditions

\begin{equation}
\overline{\rho u'_i u'_i} \geq 0,
\end{equation}

\begin{equation}
\left( \overline{\rho u'_i u'_j} \right)^2 \leq \overline{\rho u'_i u'_i} \overline{\rho u'_j u'_j}
\end{equation}

imply, in terms of the modeled turbulent stresses,

\begin{equation}
\frac{2}{3} \rho \bar{k} - 2 \bar{\mu}_i \left( S_{ii} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) \geq 0, \quad i = 1, 2, 3,
\end{equation}

\begin{equation}
\left( -2 \bar{\mu}_r S_{ij} \right)^2 \leq \left[ \frac{2}{3} \rho \bar{k} - 2 \bar{\mu}_i \left( S_{ii} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) \right] \left[ \frac{2}{3} \rho \bar{k} - 2 \bar{\mu}_j \left( S_{jj} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) \right], \quad i, j = 1, 2, 3, \quad i \neq j.
\end{equation}

From Eqs. (13) and (14), recalling Eq. (10), we obtain

\begin{equation}
\frac{e^{\bar{\omega}}}{\bar{\omega}^*} - 3 \left( S_{ii} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) \geq 0, \quad i = 1, 2, 3,
\end{equation}

\begin{equation}
\left( \frac{e^{\bar{\omega}}}{\bar{\omega}^*} \right)^2 - 3 \left( S_{ii} + S_{jj} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) \frac{e^{\bar{\omega}}}{\bar{\omega}^*} + 9 \left[ \left( S_{ii} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) \left( S_{jj} - \frac{1}{3} \frac{\bar{\partial} u_k}{\bar{\partial} x_k} \right) - S_{ij}^2 \right] \geq 0,
\end{equation}

\begin{equation}
i, j = 1, 2, 3, \quad i \neq j.
\end{equation}

Let us denote with $a$ the maximum value of the unknown $e^{\bar{\omega}}/\bar{\omega}^*$ corresponding to the zeros of Eqs. (15) and (16). Then the lower bound $\bar{\omega}_{r0}$ that guarantees realizable turbulent stresses is given by

\begin{equation}
\frac{e^{\bar{\omega}_{r0}}}{\bar{\omega}^*} = a.
\end{equation}

The solution of Eq. (17) is trivial for the high-Reynolds number $k–\omega$ model because in this case $\bar{\omega}^*$ is constant. For the low-Reynolds number $k–\omega$ model $\bar{\omega}^*$ depends on the turbulent Reynolds number according to the equation

\begin{equation}
\bar{\omega}^* = \bar{\omega}_0^* + \frac{R e_i / R_k}{1 + R e_i / R_k},
\end{equation}

where $\bar{\omega}_0^*$, $\bar{\omega}_0^*$ and $R_k$ are constants and $R e_i$ is the turbulent Reynolds number given by $R e_i = k / (e^{\bar{\omega}^*} v)$. Combining Eqs. (17) and (18) we find $\bar{\omega}_{r0}$ from the following second degree equation for the unknown $e^{\bar{\omega}_{r0}}$

\begin{equation}
e^{2\bar{\omega}_{r0}} - \left( \bar{\omega}_r^* \bar{\omega}_0^* a - \frac{k}{R_k v} \right) e^{\bar{\omega}_{r0}} - \bar{\omega}_r^* \frac{k}{R_k v} a = 0
\end{equation}
and, finally, we set $\tilde{\omega}_r$ in Eqs. (2), (4), (5) and (10) as

$$\tilde{\omega}_r = \max (\tilde{\omega}, \tilde{\omega}_\infty).$$

(20)

### 2.3. Solid wall boundary condition for $\omega$

One of the methods usually employed to prescribe the solid wall boundary condition for $\omega$ is the so-called rough-wall method proposed by Wilcox [4]. In this method both rough and smooth walls can be simulated by prescribing appropriate values for the wall surface roughness and by computing the wall value $\omega_w$ by means of the relation

$$\omega_w = S_r \frac{u_t^2}{\nu_w},$$

(21)

where $u_t = \sqrt{\tau_w/\rho_w}$ is the friction velocity and $\tau_w$, $\rho_w$ and $\nu_w$ are the shear stress, the density and the kinematic viscosity at the wall. The non-dimensional function $S_r$ given by Wilcox is defined as

$$S_r = \left\{ \begin{array}{ll} (50/k_r^+)^2 & \text{if } k_r^+ < 25, \\ 100/k_r^+ & \text{if } k_r^+ \geq 25, \end{array} \right.$$

(22)

where $k_r^+ = k_r u_t/\nu_w$ denotes the non-dimensional equivalent sand-roughness height (not to be confused with the turbulent kinetic energy $k$). For smooth wall simulations Wilcox recommends to apply Eq. (21) on “sufficiently fine grids” and to use “small values of $k_r^+$”. According to Wilcox, this means that the distance $y_1^+ = y_1 u_t/\nu_w$ of the first grid point off the wall satisfies to the relation $y_1^+ \leq 2.5$ and that $k_r^+ < 5$.

A second approach to prescribe the wall boundary condition $\omega_w$ is that proposed by Menter [8], whereby the prescribed smooth wall value $\omega_w$ is related to the distance $y_1$ according to the relation

$$\omega_w = 10 \frac{6\nu}{\beta y_1^2}.$$  

(23)

The combination of Eq. (23) and (21) in the case of smooth walls ($S_r = (50/K_r^+)^2$) indicates that Menter’s approach implies that the prescribed wall roughness should be related to the distance of the grid points nearest to the wall according to the linear relation

$$k_r = C_w y_1, \quad C_w = \sqrt{\frac{50^2}{10}} \frac{6}{\beta} = 1.77.$$ 

(24)

The above conclusion is also supported by the accurate near-wall numerical study of the $\omega$ solution and the comparison of skin friction distributions of flat plate flows computed on differently refined grids reported in [9], which clearly indicate that using Eq. (24) to prescribe wall roughness in smooth wall simulations greatly reduces the grid-dependence of the predicted skin friction distribution. It is to be noticed, however, that the numerical experiments performed by Hellsten [9] suggest an optimal value $C_w = 2.5$ whereas we have obtained optimal results from the point of view of the computed skin friction coefficient by using $C_w = 0.3$. It therefore appears that the optimal value of the coefficient $C_w$ is strongly dependent upon the numerical discretization scheme employed.
3. DG space discretization of the RANS equations

The complete set of the RANS and $k-\omega$ turbulence model equations can be written in compact form as

$$\frac{\partial u}{\partial t} + \nabla \cdot f_c(u) + \nabla \cdot f_v(u, \nabla u) + s(u, \nabla u) = 0,$$

(25)

where $u \in \mathbb{R}^{d+4}$ and $s \in \mathbb{R}^{d+4}$ denote the vectors of the conservative variables and of the source terms, $f_c$ and $f_v \in \mathbb{R}^{d+4} \setminus \mathbb{R}^d$ denote the inviscid and viscous flux functions, respectively, and $d$ is the space dimension. The entries of $u, s, f_c$ and $f_v$ can be found by comparison with Eqs. (1)–(9).

The viscous flux $f_v$ is a linear function of the gradient $\nabla u$ and Eq. (25) can therefore also be written as

$$\frac{\partial u}{\partial t} + \nabla \cdot f_c(u) + \nabla \cdot [\mathcal{A}(u) \nabla u] + s(u, \nabla u) = 0.$$

(26)

3.1. The inviscid NS equations

We will first briefly present the DG method for the space discretization of the inviscid NS (Euler) equations, which can be written as

$$\frac{\partial u}{\partial t} + \nabla \cdot f_c(u) = 0$$

(27)

with suitable initial and boundary conditions. The weak form of the above equations can be written as

$$\int_{\Omega} \frac{\partial u}{\partial t} \, d\Omega - \int_{\Omega} \nabla v \cdot f_c(u) \, d\Omega - \int_{\Gamma} v f_c(u^*) \cdot n \, d\sigma = 0 \quad \forall v.$$

(28)

The state $u^*$ appearing in the contour integral is introduced in order to weakly prescribe the boundary conditions.

In order to construct a DG discretization of system (28) we consider an approximation $\Omega_h$ of $\Omega$ and a triangulation $\mathcal{T}_h = \{e\}$ of $\Omega_h$. We denote by $\mathcal{I}_h = \{i\}$ and $\mathcal{B}_h = \{b\}$ the set of interface and boundary edges of $\mathcal{T}_h$, respectively, and by $\Sigma_h$ and $\Gamma_h$ the geometrical locii corresponding to $\mathcal{I}_h$ and $\mathcal{B}_h$. We consider piecewise polynomial functions on $\mathcal{T}_h$ with no global continuity requirement, i.e. function which are in general double valued on $\mathcal{I}_h$. If $P^k(e)$ denotes the space of polynomial functions of degree at most $k$ in element $e$, and if we consider the function space

$$V_h := \{v \in L^2(\Omega_h)^{d+2} : v|_e \in P^k(e) \ \forall e \in \mathcal{T}_h\},$$

(29)

the DG formulation of (28) is then as follows: find $u_h \in V_h$ so that

$$\sum_{e \in \mathcal{T}_h} \int_{\Omega_e} v_h \frac{\partial u_h}{\partial t} \, d\Omega - \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \nabla v_h \cdot f_c(u_h) \, d\Omega + \sum_{e \in \mathcal{T}_h} \int_{\Sigma_h} v_h f_c(u_h, u_h^*, n) \, d\sigma = 0$$

(30)

holds for an arbitrary test function $v_h \in V_h$.

The various summation appearing in Eq. (30) are mandatory since the integration by part rule cannot be applied to the entire domain $\Omega_h$ because of the discontinuities at element interfaces.
Instead, we first have to split the integral over \( \Omega_h \) into the sum of integrals over the elements \( \Omega_e \) of \( \mathcal{T}_h \) and then apply integration by parts on each (integral) term of the sum. This is a valid operation since the functions are locally continuous inside the elements of \( \mathcal{T}_h \). The “normal numerical flux function” \( \hat{f}_c(u_h, u_h^i; \mathbf{n}) \) appearing in (30) has been introduced in order to (1) uniquely define the contour integrals on \( \Sigma_h \) where \( u_h \) is double valued and (2) to prescribe in weak sense the boundary data on \( \Gamma_h \). In the first case, the function \( \hat{f}_c \) depends on the traces of \( u_h \) and \( u_h^i \) on \( \Sigma_h \) of the two elements sharing an internal interfaces (the superscript a in \( u_h^a \) indicates the “adjacent” element). Any numerical flux function used in upwind finite volume methods can be used. In this work we have considered the van Leer flux vector split formulation as modified by Hänel [10]. In the second case we simply set \( \hat{f}_c = f_c(u) \cdot \mathbf{n} \). Notice that the use of \( \hat{f}_c \) on internal interfaces introduces the coupling between the unknowns of neighbouring elements which is obviously an essential ingredient for a meaningful discretization of (28) and which would otherwise be completely missing.

Let us now rewrite the various summations appearing in system (30) as integrals over the entire domain \( \Omega_h \), the interface \( \Sigma_h \) and the boundary \( \Gamma_h \). The summations of integrals over the element domains \( \Omega_e \) are obviously equivalent to integrals over the entire domain \( \Omega_h \), while the summation of contour integrals can be rewritten as

\[
\sum_{e \in \mathcal{T}_h} \int_{\Omega_e} v_h \hat{f}_c(u_h, u_h^d; \mathbf{n}) \, d\sigma = \sum_{e \in \mathcal{T}_h} \int_{\Sigma_i} \left[ v_h \hat{f}_c(u_h^i, u_h^i; \mathbf{n}^-) + v_h \hat{f}_c(u_h^i, u_h^-; \mathbf{n}^+) \right] \, d\sigma + \sum_{b \in \partial_h} \int_{\Gamma_b} v_h \hat{f}_c(u_h^b) \cdot \mathbf{n} \, d\sigma, \tag{31}\]

where \((\cdot)^-\) and \((\cdot)^+\) denote the values of the trace on \( \Sigma_i \) of any quantity \((\cdot)\) evaluated for the two elements \( e^- \) and \( e^+ \) separated by edge \( i \), and \( \mathbf{n}^- \) and \( \mathbf{n}^+ \) are the unit vectors normal to \( i \) pointing exterior to \( e^- \) and \( e^+ \), respectively. By considering that \( \mathbf{n}^- + \mathbf{n}^+ = \mathbf{0} \) and by using the so-called “directional consistency” property \( \hat{f}_c(u, v; \mathbf{n}) = -\hat{f}_c(v, u; -\mathbf{n}) \) which is satisfied by any normal numerical flux function for arbitrary functions \( u \) and \( v \) and vector \( \mathbf{n} \), we can therefore rewrite Eq. (30) as

\[
\int_{\Omega_h} v_h \frac{\partial u_h}{\partial t} \, d\Omega - \int_{\Omega_h} \nabla v_h \cdot f_c(u_h) \, d\Omega + \int_{\Sigma_h} (v_h^i - v_h^+) \hat{f}_c(u_h^i, u_h^i; \mathbf{n}^-) \, d\sigma + \int_{\Gamma_h} v_h \hat{f}_c(u_h^b) \cdot \mathbf{n} \, d\sigma = 0 \quad \forall v_h \in V_h. \tag{32}\]

We next turn to the DG space discretization of the viscous term of the RANS equations.

### 3.2. The Helmholtz problem

DG discretization methods for the viscous part of the NS equations are generally less well known than those used for the inviscid part of the equations. This is particularly true in the case of the RANS equations where very little has been published. For this reason we will here describe the DG method employed for the discretization of the viscous part of the RANS equation in considerable detail. In order to simplify the presentation the main ideas will be first presented with
reference to the simple purely elliptic Helmholtz problem with Dirichlet and Neumann conditions in $\mathbb{R}^d$
\[
\nabla^2 u - xu = s \quad \text{in } \Omega,
\]
\[
u = u_b \quad \text{on } \Gamma^0 \in \partial \Omega, \quad \partial_n u = z^b \quad \text{on } \Gamma^a \in \partial \Omega,
\]
where $\Gamma^0 \cup \Gamma^a = \partial \Omega$. It is not clear at this stage how to accommodate the Laplacian operator in a space of purely advective problems. In order to adopt the techniques already developed in the case of purely advective problems, we therefore reformulate the Helmholtz problem as the first-order system
\[
z = \nabla u, \quad u = u_b \quad \text{on } \Gamma^0, \quad \nabla \cdot z - xu = s, \quad z \cdot n = x^b \quad \text{on } \Gamma^a,\]
where $z \in \mathbb{R}^d$, $u \in \mathbb{R}$. The weak variational form of system (34) may be written as
\[
\int_{\Omega} g \cdot z \, d\Omega = - \int_{\Omega} u \nabla \cdot g \, d\Omega + \int_{\Gamma} u^n g \cdot n \, d\sigma \quad \forall g, \quad (35a)
\]
\[
- \int_{\Omega} \nabla v \cdot z \, d\Omega + \int_{\Gamma} v z^r \cdot n \, d\sigma - x \int_{\Omega} vv \, d\Omega = \int_{\Omega} vs \, d\Omega \quad \forall v, \quad (35b)
\]
where $g \in \mathbb{R}^d$ and $v \in \mathbb{R}$ are arbitrary test functions for the first and the second equations, respectively, $u^r = u_b$ on $\Gamma^0$, $u^r = u$ on $\Gamma^a$, $z^r \cdot n = x^b$ on $\Gamma^a$, and $z^r = z$ on $\Gamma^0$.

3.2.1. Discontinuous finite element discretization

Let us consider the functions spaces
\[
V_h := \{ v \in L^2(\Omega_h) : v|_e \in P^k(e) \forall e \in \mathcal{T}_h \},
\]
\[
G_h := \{ g \in [L^2(\Omega_h)]^d : g|_e \in [P^k(e)]^d \forall e \in \mathcal{T}_h \},
\]
where $P^k(e)$ denotes as before the space of polynomial functions of degree at most $k$ in element $e$. The DG formulation of (35a) is then as follows: find $u_h \in V_h$ and $z_h \in G_h$ so that,
\[
\sum_{e \in \mathcal{T}_h} \int_{\Omega_e} g_h \cdot z_h \, d\Omega = - \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} u_h \nabla \cdot g_h \, d\Omega + \sum_{e \in \mathcal{T}_h} \int_{\partial \Omega_e} \tilde{u}_h g_h \cdot n \, d\sigma \quad (37a)
\]
and
\[
- \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \nabla v_h \cdot z_h \, d\Omega + \sum_{e \in \mathcal{T}_h} \int_{\partial \Omega_e} v_h \tilde{z}_h \cdot n \, d\sigma + \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} v_h u_h \, d\Omega = \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} v_h s \, d\Omega, \quad (37b)
\]
holds for arbitrary test functions $g_h \in G_h$ and $v_h \in V_h$. Similarly to the previously considered inviscid case, the scalar and vector “numerical flux functions” $\tilde{u}_h$ and $\tilde{z}_h$ appearing in (37a) have been introduced in order to uniquely define the contour integrals on $\Sigma_h$ where $u_h$ and $z_h$ are double valued and to prescribe in weak sense the boundary data on $\Gamma_h$.

To simplify the notation, let us now rewrite the various summations appearing in system (37a) as integrals over the entire domain $\Omega_h$, the interface $\Sigma_h$ and the boundary $\Gamma_h$. The two summation of contour integrals can in particular be written as
\[
\sum_{i \in \Gamma_h} \int_{\Sigma_i} \bar{u}_h [(g_h \cdot n)^- + (g_h \cdot n)^+] \, d\sigma + \sum_{b \in \partial \Sigma_h} \int_{\Gamma_b} \bar{u}_h g_h \cdot n \, d\sigma, \\
\sum_{i \in \Gamma_h} \int_{\Sigma_i} \bar{\zeta}_h \cdot [(v_h n)^- + (v_h n)^+] \, d\sigma + \sum_{b \in \partial \Sigma_h} \int_{\Gamma_b} v_h \bar{\zeta}_h \cdot n \, d\sigma,
\]

where \((-\cdot)^-\) and \((-\cdot)^+\) denote as before the values of the trace on \(\Sigma_i\) of any quantity evaluated for the two elements \(e^-\) and \(e^+\) which share edge \(i\). Following the ideas put forward in [2,3,11], it is convenient to introduce a “vector jump operator” \(\mathcal{J}\) acting on \(\Sigma_h\) defined as

\[
\mathcal{J}x = x^- n^- + x^+ n^+, \quad \mathcal{J} \cdot y = y^- \cdot n^- + y^+ \cdot n^+
\]

for arbitrary scalar and vector functions \(x\) and \(y\), respectively. Notice that the jump of a scalar function is vector valued while the jump of a vector valued function is scalar. \(^1\) Notice also that, because of the relation \(n^+ = -n^-\), this is indeed a jump operator despite the plus sign appearing in the definitions (39).

By virtue of the relations (38) and (39), we can therefore rewrite system (37a) and (37b) as

\[
\int_{\Omega_h} g_h \cdot z_h \, d\Omega = -\int_{\Omega_h} u_h \nabla \cdot g_h \, d\Omega + \int_{\Sigma_h} \bar{u}_h \mathcal{J} \cdot g_h \, d\sigma + \int_{\Gamma_b} u^h g_h \cdot n \, d\sigma + \int_{\Gamma_b} u_h g_h \cdot n \, d\sigma \tag{40a}
\]

and

\[
-\int_{\Omega_h} \nabla v_h \cdot z_h \, d\Omega + \int_{\Sigma_h} \bar{\zeta}_h \cdot \mathcal{J} v_h \, d\sigma + \int_{\Gamma_b} \bar{z} \cdot n v_h \, d\sigma + \int_{\Gamma_b} \bar{z} \cdot \bar{v}_h \, d\sigma - \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega, \tag{40b}
\]

where, from now on, we let understood that weak forms such as Eqs. (40a) and (40b) must hold for arbitrary test functions \(g_h \in G_h\) and \(v_h \in V_h\). To completely define the DG approximation we still have to provide a definition for the numerical flux functions \(\bar{u}_h\) and \(\bar{\zeta}_h\). This crucial point is the topic of the following sections.

### 3.2.2. First equation

We begin with the first equation of system (40a) and consider the “centered” numerical flux \(\bar{u}_h = \{u_h\}\), where \(\{\cdot\}\) denotes the average or the identity operator for internal or boundary edges, respectively, i.e.

\[
\{(-\cdot)^-\} = \left\{ \begin{array}{ll}
\frac{1}{2} (-\cdot)^- + (-\cdot)^+ & \text{on } \Sigma_h, \\
(-\cdot)^- & \text{on } \Gamma_b.
\end{array} \right.
\]

This is a very natural choice for the discretization of a purely elliptic problem. With the above choice for \(\bar{u}_h\), the first equation of system (40a) may be rewritten as

\(^1\) The notation \(\mathcal{J}\) and \(\mathcal{J} \cdot\) has been preferred to that used in [2,3,11] since it reflects the vector nature of the jump operator and because its usage is analogous to that of the well known \(\nabla\) and \(\nabla\cdot\) operators.
\[
\int_{\Omega_h} \mathbf{g}_h \cdot \mathbf{z}_h \, d\Omega = - \int_{\Omega_h} u_h \mathbf{V} \cdot \mathbf{g}_h \, d\Omega + \int_{\Sigma_h} \{u_h\} \mathbf{J} \cdot \mathbf{g}_h \, d\sigma + \int_{\Gamma_h^0} u_h \mathbf{g}_h \cdot \mathbf{n} \, d\sigma + \int_{\Omega_h} u_h \mathbf{g}_h \cdot \mathbf{n} \, d\sigma. \tag{42}
\]

A more convenient form of (42) can be obtained by considering the identity
\[
\int_{\Omega_h} (x_h \mathbf{V} \cdot \mathbf{y}_h + \mathbf{y}_h \cdot \nabla x_h) \, d\Omega = \int_{\Sigma_h} \left[ \{x_h\} \mathbf{J} \cdot \mathbf{y}_h + \{y_h\} \cdot \mathbf{J} x_h \right] \, d\sigma + \int_{\Gamma_h} x_h \mathbf{y}_h \cdot \mathbf{n} \, d\sigma, \tag{43}
\]
which holds for arbitrary \( x_h \in V_h \) and \( \mathbf{y}_h \in G_h \) and is a consequence of the divergence theorem and of the relation
\[
(x_h \mathbf{y}_h \cdot \mathbf{n})^- + (x_h \mathbf{y}_h \cdot \mathbf{n})^+ = \{x_h\} \mathbf{J} \cdot \mathbf{y}_h + \{y_h\} \cdot \mathbf{J} x_h,
\]
which can be verified by simple algebraic calculation. We have in fact that
\[
\int_{\Omega_h} (x_h \mathbf{V} \cdot \mathbf{y}_h + \mathbf{y}_h \cdot \nabla x_h) \, d\Omega = \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} (x_h \mathbf{V} \cdot \mathbf{y}_h + \mathbf{y}_h \cdot \nabla x_h) \, d\Omega
\]
\[
= \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \mathbf{V} \cdot (x_h \mathbf{y}_h) \, d\Omega = \sum_{e \in \mathcal{T}_h} \int_{\Gamma_{e}} x_h \mathbf{y}_h \cdot \mathbf{n} \, d\sigma
\]
\[
= \sum_{i \in \mathcal{I}_h} \int_{\Sigma_i} \left[ (x_h \mathbf{y}_h \cdot \mathbf{n})^- + (x_h \mathbf{y}_h \cdot \mathbf{n})^+ \right] \, d\sigma + \sum_{b \in \mathcal{B}_h} \int_{\Gamma_b} x_h \mathbf{y}_h \cdot \mathbf{n} \, d\sigma
\]
\[
= \int_{\Sigma_h} \left[ \{x_h\} \mathbf{J} \cdot \mathbf{y}_h + \{y_h\} \cdot \mathbf{J} x_h \right] \, d\sigma + \int_{\Gamma_h} x_h \mathbf{y}_h \cdot \mathbf{n} \, d\sigma. \tag{44}
\]
If we replace the first integral on the right-hand side of (42) using identity (43) with \( x_h \) and \( \mathbf{y}_h \) replaced by \( u_h \) and \( \mathbf{g}_h \), respectively, i.e.
\[
- \int_{\Omega_h} u_h \mathbf{V} \cdot \mathbf{g}_h \, d\Omega = \int_{\Omega_h} \mathbf{g}_h \cdot \nabla u_h \, d\Omega - \int_{\Sigma_h} \{u_h\} \mathbf{J} \cdot \mathbf{g}_h + \{\mathbf{g}_h\} \cdot \mathbf{J} u_h \, d\sigma - \int_{\Gamma_h} u_h \mathbf{g}_h \cdot \mathbf{n} \, d\sigma, \tag{45}
\]
we obtain an equivalent form of Eq. (40a) which reads as
\[
\int_{\Omega_h} \mathbf{g}_h \cdot \mathbf{z}_h \, d\Omega - \int_{\Omega_h} \mathbf{g}_h \cdot \nabla u_h \, d\Omega + \int_{\Sigma_h} \{\mathbf{g}_h\} \cdot \mathbf{J} u_h \, d\sigma + \int_{\Gamma_h^0} \mathbf{g}_h \cdot \mathbf{n} u_h \, d\sigma = \int_{\Gamma_h^0} \mathbf{g}_h \cdot \mathbf{n} \, d\sigma. \tag{46}
\]
Notice that the boundary integral on \( \Gamma_h^0 \) of Eq. (42) disappears from Eq. (46), and that only the integral over \( \Gamma_h^0 \) survives. This observation suggests to adopt an “extended” jump operator \( \mathbf{J}^0 \) which automatically takes care of Dirichlet boundary conditions defined as
\[
\mathbf{J}^0 \mathbf{x} = \begin{cases} (x - x^b) \mathbf{n} & \text{on } \Gamma_h^0, \\ \mathbf{x}^- \mathbf{n}^- + x^+ \mathbf{n}^+ & \text{on } \Sigma_h, \end{cases} \tag{47}
\]
which allows to compactly rewrite Eq. (46) as
\[
\int_{\Omega_h} \mathbf{g}_h \cdot \mathbf{z}_h \, d\Omega - \int_{\Omega_h} \mathbf{g}_h \cdot \nabla u_h \, d\Omega + \int_{\Sigma_h^0} \{\mathbf{g}_h\} \cdot \mathbf{J}^0 u_h \, d\sigma = 0, \tag{48}
\]
where \( \Sigma_h^0 = \Sigma_h \cup \Gamma_h^0 \). In the next section we consider the second equation of the system.
3.2.3. Second equation and the BR1 scheme

The simplest—and apparently more natural—choice for the vector flux $\mathbf{z}_h$ is again the average $\mathbf{z}_h = \{\mathbf{z}_h\}$. With this choice, the weak variational DG discretization of system (34) can be written as

$$
\int_{\Omega_h} g_h \cdot \mathbf{z}_h \; d\Omega - \int_{\Gamma_0_h} g_h \cdot \mathbf{v}_h \; d\Gamma + \int_{\Sigma_h} \{g_h\} \cdot \mathbf{S}_0 u_h \; d\sigma = 0
$$

and

$$
- \int_{\Omega_h} \nabla v_h \cdot \mathbf{z}_h \; d\Omega - \int_{\Sigma_h} \{z\} \cdot \mathbf{S}_0 \mathbf{v}_h \; d\sigma - \alpha \int_{\Omega_h} v_h u_h \; d\Omega = \int_{\Omega_h} v_h S \; d\Omega - \int_{\Gamma_h} \mathbf{z}_h^T v_h \; d\sigma,
$$

where we have assumed that the test function on $\Gamma_0_h$ satisfies the homogeneous Dirichlet boundary data $v^b = 0$ on $\Gamma_0_h$. This DG method was first introduced in [12] for the discretization of the viscous terms of the Navier–Stokes equations, and, following the naming convention subsequently introduced in [3], will be called “BR1 scheme”.

Despite the promising performance displayed by the method in several test calculation performed, subsequent numerical experiments and theoretical investigations showed that the scheme has some limitations when applied to purely elliptic problems such as those considered in this section, in that it is not optimally accurate for piecewise polynomial approximations of odd order and it cannot be used for the case of a purely Poisson problem, i.e. it is unsuitable if the coefficient $\alpha = 0$ in (33). Another drawback of the BR1 scheme—probably the most serious one for the practical applicability of the method—is the need to introduce the auxiliary variable $\mathbf{z}_h$, which imply additional computational cost with respect to a method formulated entirely in terms of the primal unknown $u_h$. In practice, the first equation of the system can used to obtain explicitly $\mathbf{z}_h$ in terms of $u_h$ at the cost of a relatively inexpensive inversion of a block diagonal mass matrix. By using this expression of the auxiliary variable in the second equation of the system, a formulation for the primal variable $u_h$ alone is obtained. However, the primal unknown $u_h$ for any (internal) element $e$ is coupled in this case not only with the unknowns of the neighbouring elements (defined as before as the elements which share an edge with element $e$) but also with the unknowns associated to the neighbours of the neighbours (this sort of things happen in general when a second-order derivative is evaluated as “the first derivative of a first derivative”). The “enlarged stencil” of the scheme implies again substantial additional computational cost (both in terms of storage and in computational time) in comparison with a compact scheme for the primal unknown $u_h$ only.

Several alternative formulations which overcome the above mentioned limitations have been devised, see e.g. [3]. Among the many method described in the literature, we here describe a scheme originally introduced by the authors in the context of the numerical solution of the RANS equations [1] which can be regarded as a modified BR1 scheme, and which has been named in [3] “BR2 scheme”. In order to present the BR2 scheme, however, we will first reformulate the original BR1 scheme in a way which turns out to be a more convenient starting point for the derivation of the BR2 scheme.

3.2.4. Lift operators

The BR2 scheme was obtained in an attempt to construct a “compact” DG approximation for a purely elliptic problem, i.e. an approximation in which the primal unknown $u_h$ for an
arbitrary element $e$ is coupled only with the unknowns of the adjacent elements. We will therefore first put in evidence the terms which are responsible for the non-compact support of the BR1 scheme.

As a matter of fact, if we express $z_h$ as a linear function of $u_h$ using Eq. (49a) and substitute this expression in Eq. (49b), there is only one term responsible for the non-compact support of the BR1 scheme, namely the interface integral containing the average $\{z_h\}$ appearing in Eq. (49b). The domain integral containing the term $\nabla v_h z_h$ of Eq. (49b) is instead not a problem in this respect. Notice in fact that for the special case $g_h = v_h$ Eq. (49a) reduces to

$$\int_{\Omega_h} V v_h \cdot z_h \, d\Omega = \int_{\Omega_h} V v_h \cdot V u_h \, d\Omega - \int_{\Sigma^0_h} \{V v_h\} \cdot J^0 u_h \, d\sigma$$

and that, upon substitution of the previous expression in Eq. (49b), we obtain

$$- \int_{\Omega_h} V v_h \cdot V u_h \, d\Omega + \int_{\Sigma^0_h} \{V v_h\} \cdot J^0 u_h + \{z\} \cdot J^0 v_h \, d\sigma$$

$$+ \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{R_h} z^h v_h \, d\sigma,$$

which is an alternative form of Eq. (49b) showing that only the trace on $\Sigma^0_h$ of the auxiliary variables $z_h$ really matters in the BR1 scheme (actually only the normal component of $z_h$).

More insight in the structure of the DG discretized problem can be obtained by inspection of Eq. (49a), which shows that the auxiliary variable $z_h$ is equal to the "internal" gradient $V u_h$ plus an additional term accounting for the jumps in $u_h$ occurring at the element interfaces and the jumps between $u_h$ and $u^b$ occurring at the boundaries where Dirichlet conditions are prescribed. This interpretation suggests to rewrite Eq. (49a) as

$$\int_{\Omega_h} g_h \cdot (z_h - V u_h) \, d\Omega = - \int_{\Sigma^0_h} \{g_h\} \cdot J^0 u_h \, d\sigma$$

and to introduce a so-called "lift" operator

$$\int_{\Omega_h} g_h \cdot R_h (J^0 u_h) \, d\Omega = - \int_{\Sigma^0_h} \{g_h\} \cdot J^0 u_h \, d\sigma,$$

which is a function that "extends" to the entire domain $\Omega_h$ the effects of the interface and boundary jumps in $u_h$. By virtue of the lift operator $R_h$, we can then express the variable $z_h$ in weak or variational sense as

$$\int_{\Omega_h} g_h \cdot z_h \, d\Omega = \int_{\Omega_h} g_h \cdot V u_h \, d\Omega + \int_{\Omega_h} g_h \cdot R_h (J^0 u_h) \, d\Omega.$$

The above relation implies that, $v_h$ and $g_h$ being arbitrary functions, we can also express both $z_h$ and $\hat{z}_h$ in "finite form" as

$$z_h = V u_h + R_h (J^0 u_h), \quad \hat{z}_h = \{z_h\} = \{V u_h\} + \{R_h (J^0 u_h)\}.$$
By substituting the above expressions into Eq. (49b) we obtain

\[- \int_{\Omega_h} \mathbf{v}_h \cdot [\nabla u_h + \mathbf{R}_h(\mathbf{f}^0 u_h)] \, d\Omega + \int_{\Sigma_h^0} \{ \nabla u_h + \mathbf{R}_h(\mathbf{f}^0 u_h) \} \cdot \mathbf{f}^0 v_h \, d\sigma \]

or, if we instead consider Eq. (50),

\[- \int_{\Omega_h} \mathbf{v}_h \cdot \nabla u_h \, d\Omega + \int_{\Sigma_h^0} [\{ \nabla v_h \} \cdot \mathbf{f}^0 u_h + \{ \nabla u_h \} \cdot \mathbf{f}^0 v_h] \, d\sigma + \int_{\Sigma_h^0} \{ \mathbf{R}_h(\mathbf{f}^0 u_h) \} \cdot \mathbf{f}^0 v_h \, d\sigma \]

These expressions show that it is only the “jump” contribution \( \mathbf{R}_h(\mathbf{f}^0 u_h) \) to the auxiliary variable \( z_h \) which is responsible for the non-compact support of the BR1 scheme, and suggests that, by modifying the definition of the lift operator, we could arrive at a scheme with the sought for compact support.

3.2.5. The BR2 scheme

The appearance of the average lift \( \mathbf{R}_h \) in the definition (53) of the vector flux implies that, for an arbitrary edge \( \sigma \), the numerical vector flux \( \mathbf{z}_h \) depends not only from the jump of \( u_h \) on that edge but also from the jumps on all the edges belonging to the elements \( e^- \) and \( e^+ \) which share edge \( \sigma \). A vector flux definition which follows more closely the numerical flux function ideas usually employed in FV schemes would instead accommodate only for the jump of \( u_h \) associated to a single edge \( \sigma \). A typical treatment of an elliptic operator in FV schemes is in fact based on the definition of auxiliary “staggered” control volumes enclosing the boundaries of the primal control volumes that are used to construct “diffusive flux terms” which are the analogue of the vector flux \( \mathbf{z}_h \).

This observation is the guideline for the following modification of the previously described scheme. Let us first consider “local” lift operators \( \mathbf{r}_h^\sigma \) for each interface or boundary edge \( \sigma \in \Sigma_h^0 \) defined as

\[ \int_{\Omega_{h\sigma}^i} \mathbf{g}_h \cdot \mathbf{r}_h^\sigma(\mathbf{f}^0 u_h) \, d\Omega = - \int_{\sigma} \{ \mathbf{g}_h \} \cdot \mathbf{f}^0 u_h \, d\sigma \quad \forall \sigma \in \Sigma_h^0 \quad \forall \mathbf{g}_h \in \mathbf{G}_h, \]  

where \( \Omega_{h\sigma}^i \) is the union of the two (or one) elements sharing the interface (or boundary) edge \( \sigma \). Notice that we therefore define a number of local lifts equal to the number of interface plus the number of Dirichlet-boundary edges, and that each local lift is non-null only on its domain \( \Omega_{h\sigma}^i \). The local lifts \( \mathbf{r}_h^\sigma \) may simply be regarded as interface (and boundary) contributions to the previously considered global lift \( \mathbf{R}_h \). By combining the definitions (52) and (56) we have in fact that
\[
\int_{\Omega_h} g_h \cdot R_h(\mathcal{J}^0 u_h) \, d\Omega = - \int_{\Sigma_h} \{g_h\} \cdot \mathcal{J}^0 u_h \, d\sigma = - \sum_{\sigma \in \Sigma_h^0} \int_{\sigma} \{g_h\} \cdot \mathcal{J}^0 u_h \, d\sigma = \sum_{\sigma \in \Sigma_h^0} \int_{\Omega_h^\sigma} g_h \cdot r^\sigma_h(\mathcal{J}^0 u_h) \, d\Omega
\]  
(57)

or, since \( g_h \) is an arbitrary function,
\[
R(\mathcal{J}^0 u_h) = \sum_{\sigma \in \Sigma_h^0} r^\sigma_h(\mathcal{J}^0 u_h).
\]

The BR2 scheme is obtained by replacing the global lift \( R_h \) appearing in the interface integral of Eq. (54) or (55) with the local lifts \( r^\sigma_h \) (notice that the global lift \( R_h \) is instead maintained in the domain integral of Eq. (54)), thus obtaining
\[
- \int_{\Omega_h} \nabla v_h \cdot \left[ \nabla u_h + R_h(\mathcal{J}^0 u_h) \right] \, d\Omega + \int_{\Sigma_h^0} \{\nabla u_h\} \cdot \mathcal{J}^0 v_h \, d\sigma + \sum_{\sigma \in \Sigma_h^0} \int_{\sigma} \{r^\sigma_h(\mathcal{J}^0 u_h)\} \cdot \mathcal{J}^0 v_h \, d\sigma \\
- \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^n} z^b v_h \, d\sigma,
\]
(58)
if we consider Eq. (54), or
\[
- \int_{\Omega_h} \nabla v_h \cdot \nabla u_h \, d\Omega + \int_{\Sigma_h^0} \{\nabla v_h\} \cdot \mathcal{J}^0 u_h + \{\nabla u_h\} \cdot \mathcal{J}^0 v_h \, d\sigma + \sum_{\sigma \in \Sigma_h^0} \int_{\sigma} \{r^\sigma_h(\mathcal{J}^0 u_h)\} \cdot \mathcal{J}^0 v_h \, d\sigma \\
- \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^n} z^b v_h \, d\sigma,
\]
(59)
if we instead consider Eq. (55). By comparison with Eq. (50) we see that the numerical flux corresponding to Eqs. (58) and (59) can be written as
\[
\tilde{\xi}_h|_\sigma = \{\nabla u_h\}|_\sigma + \{r^\sigma_h(\mathcal{J}^0 u_h)\}.
\]
(60)

3.2.6. Vector average and scalar jump operators

An alternative formulation of the various DG discretizations of the Helmholtz problem considered so far can be obtained by introducing a “normal vector average” operator \( \{y_h\}_n \) which operates on vectors \( y_h \in \mathbf{G}_h \) and is defined as
\[
\{y_h\}_n = \begin{cases} 
\frac{1}{2} \left( (y_h \cdot n)^- - (y_h \cdot n)^+ \right) & \text{on } \Sigma_h, \\
y_h \cdot n & \text{on } \Gamma_h,
\end{cases}
\]
(61)
and a “scalar jump operator”
\[
\mathcal{J}^0 x_h = \begin{cases} 
x_h^- - x_h^+ & \text{on } \Sigma_h, \\
x_h - x_h^- & \text{on } \Gamma_h,
\end{cases}
\]
(62)
which operates on functions \( x_h \in \mathbf{V}_h \). Notice that, because of the relation \( n^+ = -n^- \), the operator \( \{y_h\}_n \) is indeed an average despite the minus sign in the definition (61). It is easy to verify by simple algebraic calculation that the identity
\{y_h\} \cdot \mathcal{J}^0 x_h = \{y_h\}_n \mathcal{J}^0 x_h

holds for arbitrary functions \(x_h \in V_h\) and \(y_h \in G_h\). Notice that expressions of the type \(\{y_h\} \cdot \mathcal{J}^0 x_h\) appears over and over in the various DG formulations of the Helmholtz problem considered so far. If we for example substitute in Eqs. (58) and (59) each occurrence of such terms with the equivalent form \(f y_h \cdot h g/C1 J^0 x_h\), we obtain

\[- \int_{\Omega_h} \nabla v_h \cdot [\nabla u_h + R_h(\mathcal{J}^0 u_h)] \, d\Omega + \int_{\Sigma_h^0} \{\nabla u_h\}_n \mathcal{J}^0 v_h \, d\sigma + \sum_{\sigma \in \Sigma_h^0} \int_{\sigma} \{r_h^\sigma(\mathcal{J}^0 u_h)\}_n \mathcal{J}^0 v_h \, d\sigma \]

\[- \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^n} z^h v_h \, d\sigma, \tag{63}\]

if we consider Eq. (58), and

\[- \int_{\Omega_h} \nabla v_h \cdot \nabla u_h \, d\Omega + \int_{\Sigma_h^0} \{\nabla v_h\}_n \mathcal{J}^0 u_h + \{\nabla u_h\}_n \mathcal{J}^0 v_h \} \, d\sigma + \sum_{\sigma \in \Sigma_h^0} \int_{\sigma} \{r_h^\sigma(\mathcal{J}^0 u_h)\}_n \mathcal{J}^0 v_h \, d\sigma \]

\[- \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^n} z^h v_h \, d\sigma, \tag{64}\]

if we instead consider Eq. (59). Notice that in Eq. (64) only the normal component of the local lift \(r_h^\sigma \cdot n\) on \(\Sigma_h\) appears and has therefore to be calculated (this does not instead apply to Eq. (63) because of the appearance of the global lift \(R_h\)).

### 3.3. The viscous term of the RANS equations

We will illustrate the extension of the preceding schemes to the case of the viscous part of the RANS equations with reference to the “generalized” Helmholtz problem in \(\mathbb{R}^d\) (\(d \leq 3\) in practice)

\[\nabla \cdot (\mathcal{A} \nabla u) - \alpha u = s \quad \text{in } \Omega,\]

\[u = u^b \quad \text{on } \Gamma^0 \cup \partial \Omega, \quad (\mathcal{A} \nabla u) \cdot n = f^b \quad \text{on } \Gamma^n \subseteq \partial \Omega, \tag{65}\]

where \(\Gamma^0 \cup \Gamma^n = \partial \Omega\). The model problem may be reformulated either as the first-order system

\[
\begin{cases}
\mathcal{A} \in \mathbb{R}^{d \times d}, \quad z \in \mathbb{R}^d, \quad u \in \mathbb{R}, \\
\begin{cases}
\mathcal{A} \mathcal{J} u - \alpha u = s, & (\mathcal{A} \mathcal{J} z) \cdot n = f^b \quad \text{on } \Gamma^n, \\
\mathcal{J} \in \mathbb{R}^d, \quad z \in \mathbb{R}^d, \quad u \in \mathbb{R}, \quad \text{or as the first-order system} \\
\begin{cases}
f = \mathcal{A} \mathcal{J} u, & u = u^b \quad \text{on } \Gamma^0, \\
\mathcal{A} \mathcal{J} f - \alpha u = s, & f \cdot n = f^b \quad \text{on } \Gamma^n.
\end{cases}
\end{cases}
\end{cases}
\tag{66}\]

where \(f \in \mathbb{R}^d\). We here concentrate on the DG discretization of system (66). For more information on DG schemes constructed on the basis of system (67), see e.g. \[5,13\].

The weak variational form of system (66) may be written as

\[\int_{\Omega} g \cdot z \, d\Omega = - \int_{\Omega} u \nabla \cdot g \, d\Omega + \int_{\Gamma} u^g \cdot n \, d\sigma \quad \forall g, \tag{68a}\]
where $g \in \mathbb{R}^d$ and $v \in \mathbb{R}$ are arbitrary test functions for the first and the second equations, respectively, $u^c = u^b$ on $\Gamma^0$, $u' = u$ on $\Gamma^u$, $(\mathcal{A} z)^* \cdot n = f^b$ on $\Gamma^u$, and $(\mathcal{A} z) = f$ on $\Gamma^0$.

In the following, we will extend to the case of the generalized Helmoltz problem the DG schemes already described for the case of the simple Helmoltz problem. Only the main steps of the derivation will be repeated here since the generalization is straightforward.

The DG formulation of system (68a) and (68b) can be written as

$$\sum_{e \in \mathcal{E}_h} \int_{\Omega_e} g_h \cdot z_h \, d\Omega = - \sum_{e \in \mathcal{E}_h} \int_{\Omega_e} u_h \nabla \cdot g_h \, d\Omega + \sum_{e \in \mathcal{E}_h} \oint_{\Sigma_e} \hat{u}_h g_h \cdot n \, d\sigma$$

and

$$- \sum_{e \in \mathcal{E}_h} \int_{\Omega_e} \nabla v_h \cdot (\mathcal{A} z_h) \, d\Omega + \sum_{e \in \mathcal{E}_h} \oint_{\Sigma_e} v_h (\hat{\mathcal{A}} h z_h) \cdot n \, d\sigma + \sum_{e \in \mathcal{E}_h} \int_{\Omega_e} v_h u_h \, d\Omega = \sum_{e \in \mathcal{E}_h} \int_{\Omega_e} v_h \delta \, d\Omega,$$

where $u_h \in V_h$ and $f_h \in G_h$ are unknown functions, and $v_h \in V_h$ and $g_h \in G_h$ are arbitrary test functions. The spaces $V_h$ and $G_h$ are the higher dimensional counterparts of those defined in Eq. (36).

By rewriting the various summations appearing in system (69a) as integrals over the entire domain $\Omega$, the interface $\Sigma_h$ and the boundary $\Gamma_h$, the DG formulation (69a) can be rewritten as

$$\int_{\Omega_h} g_h \cdot z_h \, d\Omega = - \int_{\Omega_h} u_h \nabla \cdot g_h \, d\Omega + \int_{\Sigma_h} \hat{u}_h \mathcal{J} \cdot g_h \, d\sigma + \int_{\Gamma^0_h} u^b g_h \cdot n \, d\sigma + \int_{\Gamma^u_h} u_h g_h \cdot n \, d\sigma$$

and

$$- \int_{\Omega_h} \nabla v_h \cdot (\mathcal{A} z_h) \, d\Omega + \int_{\Sigma_h} (\hat{\mathcal{A}} h z_h) \cdot \mathcal{J} v_h \, d\sigma + \int_{\Gamma^0_h} v_h (\mathcal{A} z_h) \cdot n \, d\sigma + \int_{\Gamma^u_h} v_h f^b \, d\sigma$$

$$- \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h \delta \, d\Omega,$$

where the $\mathcal{J}$ and $\mathcal{J} \cdot$ are defined as in (39). If we choose $\hat{u}_h = \{u_h\}$, the first equation of the system is identical to that described in the case of the simple Helmoltz problem. The auxiliary variable $z_h$ can therefore be written as

$$z_h = \nabla u_h + R_h(\mathcal{J}^0 u_h), \quad \int_{\Omega_h} g_h \cdot R_h(\mathcal{J}^0 u_h) \, d\Omega = - \int_{\Sigma_h} \{g_h\} \cdot \mathcal{J}^0 u_h \, d\sigma,$$

where the operator $\mathcal{J}^0$ is defined as in (47). By setting $(\hat{\mathcal{A}} h z_h) = \{\mathcal{A} z_h\}$, the second equation of the system can instead be obtained from the equations for the simple Helmoltz problem by substituting each occurrence of the auxiliary variable $z_h$ with the product $\hat{\mathcal{A}} h z_h$, or, because of Eq. (71), each occurrence of $\nabla u_h$ and $R_h$ with $\hat{\mathcal{A}} h \nabla u_h$ and $\hat{\mathcal{A}} h R_h$, respectively. The BR1 formulation corresponding to Eqs. (54) and (55) therefore generalize to
\[ - \int_{\Omega_h} \nabla v_h \cdot \left[ \mathcal{A}_h \left( \nabla u_h + R_h (\mathcal{J}^0 u_h) \right) \right] \, d\Omega + \int_{\Sigma_h^0} \left\{ \mathcal{A}_h \left( \nabla u_h + R_h (\mathcal{J}^0 u_h) \right) \right\} \cdot \mathcal{J}^0 v_h \, d\sigma \\
- \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^b} \zeta^b v_h \, d\sigma \] (72)

and,

\[ - \int_{\Omega_h} \nabla v_h \cdot \left( \mathcal{A}_h \nabla u_h \right) \, d\Omega + \int_{\Sigma_h^0} \left[ \left\{ \mathcal{A}_h^T \nabla v_h \right\} \cdot \mathcal{J}^0 u_h + \left\{ \mathcal{A}_h \nabla u_h \right\} \cdot \mathcal{J}^0 v_h \right] \, d\sigma \\
+ \int_{\Sigma_h^0} \left\{ \mathcal{A}_h R_h (\mathcal{J}^0 u_h) \right\} \cdot \mathcal{J}^0 v_h \, d\sigma - \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^b} \zeta^b v_h \, d\sigma, \] (73)

respectively. To obtain Eq. (73) we have used the relation

\[ \int_{\Omega_h} \nabla v_h \cdot \left( \mathcal{A}_h R_h (\mathcal{J}^0 u_h) \right) \, d\Omega = \int_{\Omega_h} \left( \mathcal{A}_h^T \nabla v_h \right) \cdot R_h (\mathcal{J}^0 u_h) \, d\Omega = - \int_{\Sigma_h^0} \left\{ \mathcal{A}_h^T \nabla v_h \right\} \cdot \mathcal{J}^0 u_h \, d\sigma \]

in which the second equality is a particular case of the right Eq. (71) for the particular choice $g_h = \mathcal{A}_h^T \nabla v_h$.

Once again the BR2 formulation can be obtained by replacing the occurrence of the global $R_h$ in the interface integrals of Eq. (72) or of Eq. (73) with the local $r_h^0$ defined in Eq. (56). The BR2 scheme for the generalized Helmholtz problem can therefore be written as

\[ - \int_{\Omega_h} \nabla v_h \cdot \left[ \mathcal{A}_h \left( \nabla u_h + R_h (\mathcal{J}^0 u_h) \right) \right] \, d\Omega + \sum_{\sigma \in \Sigma_h^0} \int_{\Sigma_h^0} \left\{ \mathcal{A}_h \left( \nabla u_h + r_h^0 (\mathcal{J}^0 u_h) \right) \right\} \cdot \mathcal{J}^0 v_h \, d\sigma \\
- \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^b} \zeta^b v_h \, d\sigma, \] (74)

or

\[ - \int_{\Omega_h} \nabla v_h \cdot \left( \mathcal{A}_h \nabla u_h \right) \, d\Omega + \sum_{\sigma \in \Sigma_h^0} \int_{\Sigma_h^0} \left[ \left\{ \mathcal{A}_h^T \nabla v_h \right\} \cdot \mathcal{J}^0 u_h + \left\{ \mathcal{A}_h \nabla u_h \right\} \cdot \mathcal{J}^0 v_h \right] \, d\sigma \\
+ \sum_{\sigma \in \Sigma_h^0} \int_{\Sigma_h^0} \left\{ \mathcal{A}_h r_h^0 (\mathcal{J}^0 u_h) \right\} \cdot \mathcal{J}^0 v_h \, d\sigma - \alpha \int_{\Omega_h} v_h u_h \, d\Omega = \int_{\Omega_h} v_h s \, d\Omega - \int_{\Gamma_h^b} \zeta^b v_h \, d\sigma. \] (75)

### 3.4 RANS equations

The DG space discretized RANS equations can be finally written as

\[ \int_{\Omega_h} v_h \frac{\partial u_h}{\partial t} \, d\Omega - \int_{\Omega_h} \nabla v_h \cdot f_c (u_h) \, d\Omega + \int_{\Sigma_h} (v_h^+ - v_h^-) \hat{f}_c (u_h^+, u_h^-; n^-) \, d\sigma + \int_{\Gamma_h} v_h f_c (u_h^+) \cdot n \, d\sigma \\
- \int_{\Omega_h} \nabla v_h \cdot \left[ \mathcal{A}_h \left( \nabla u_h + R_h (\mathcal{J}^0 u_h) \right) \right] \, d\Omega + \sum_{\sigma \in \Sigma_h^0} \int_{\Sigma_h^0} \left\{ \mathcal{A}_h \left( \nabla u_h + r_h^0 (\mathcal{J}^0 u_h) \right) \right\} \cdot \mathcal{J}^0 v_h \, d\sigma \\
+ \int_{\Omega_h} v_h (u_h, \nabla u_h + R_h (\mathcal{J}^0 u_h)) \, d\Omega = 0. \] (76)
where we have used the formulation (74) to discretize in space the viscous part of the RANS equations. The choice of Eq. (74) instead of Eq. (75) is due to the presence of a source term which is a function of both \( u_h \) and \( \nabla u_h \). This term could be evaluated simply in terms of the “internal” gradient \( \nabla u_h \), but, in the sake of robustness and consistency with the previous derivation of the DG space discretization of a diffusive operator, we preferred to evaluate it with a gradient which includes the contribution due to the jumps of \( u_h \) (and to the Dirichlet boundary conditions).

A more compact formulation of the DG semidiscretized equation is obtained by using the normal average and scalar jump operators. By formally replacing \( f/C_1 g \) with \( f/C_1 g_n \) and \( J^0 \) with \( J^0 \), respectively, (and by removing the scalar product between the two aforementioned operators) we in fact obtain after rearranging the terms

\[
\int_{\Omega_h} v_h \frac{\partial u_h}{\partial t} \, d\Omega - \int_{\Omega_h} \nabla v_h \cdot f_h(u_h, u_h^a, \nabla u_h) \, d\Omega + \int_{\Sigma_h} \hat{f}_h(u_h^+, \nabla u_h^+; n^+) J^0 v_h \, d\sigma \\
+ \int_{\Omega_h} v_h s(u_h, \nabla u_h + R_h(J^0(u_h))) \, d\Omega = 0,
\]

(77)

where \( u_h^a \) denotes the value of \( u_h \) on the adjacent elements, and the RANS flux \( f_h \) and normal numerical flux \( \hat{f}_h \) are given by

\[
\begin{align*}
\quad & f_h(u_h, u_h^a, \nabla u_h) = f_c(u_h) + \mathcal{A}_h(\nabla u_h + R_h(J^0 u_h)), \\
\quad & \hat{f}_h(u_h^+, \nabla u_h^+; n^+) = \hat{f}_c(u_h^-) + \mathcal{A}_h(\nabla u_h + R_h(J^0 u_h))^n.
\end{align*}
\]

4. Time integration

The coupled system of the DG space discretized RANS and \( k-\omega \) equations can be written as

\[
M \frac{dU}{dt} + R(U) = 0,
\]

(78)

where \( M \), \( U \) and \( R \) denote the block diagonal mass matrix, the global solution vector, and the residual vector, respectively. The solution of Eq. (77) is advanced in time be means of the second-order accurate implicit Runge–Kutta scheme proposed by Iannelly and Baker [14], which can be written as

\[
U^{n+1} - U^n = Y_1 K_1 + Y_2 K_2,
\]

(79)

\[
\left[ \frac{M}{\Delta t} + \frac{\partial R(U^n)}{\partial U} \right] K_1 + R(U^n) = 0,
\]

(80)

\[
\left[ \frac{M}{\Delta t} + \frac{\partial R(U^n)}{\partial U} \right] K_2 + R(U^n + \beta K_1) = 0.
\]

(81)

The values of the constants \( \alpha, \beta, Y_1 \) and \( Y_2 \) corresponding to an optimally second-order accurate scheme are
\[ \alpha = \frac{2 - \sqrt{2}}{2}, \quad \beta = 8\alpha \left( \frac{1}{2} - \alpha \right), \quad Y_1 = 1 - \frac{1}{8\alpha}, \quad Y_2 = 1 - Y_1. \]  \tag{82}

The backward Euler scheme is obtained by selecting \( Y_1 = 1, \ Y_2 = 0 \) and \( \alpha = 1 \). The Crank–Nicolson schemes is instead obtained for \( Y_1 = 1, \ Y_2 = 0, \ \alpha = 1/2 \). Both Eqs. (80) and (81) require the solution of a system of linear algebraic equations which can be written as \( Ax + b = 0 \).

The matrix \( A \) can be regarded as a \( n \times n \) block sparse matrix, where \( n \) denotes the number of elements \( e \) in \( \mathcal{T}_h \). The blocks are \( m \times m \) matrices, where \( m \) is the number of fields to be computed (i.e. \( \rho, p, \rho u, \rho v, \rho k, \rho \omega \)) times the number of expansion coefficients of the polynomials \( P^k \) used to approximate the solution. The number of non-zero blocks for each (block) row \( i \) of the matrix \( A \) is at most equal to the number of elements surrounding element \( i \) plus one. As reported in [15], all the Jacobian matrices are computed analytically and take into account the full dependence of the fluxes and of the source term on the unknown \( u_h \), on its gradient \( \nabla u_h \) and on the functions \( \mathcal{R}_h(\mathcal{J}^0 u_h) \) and \( \mathcal{R}_h^0(\mathcal{J}^0 u_h) \). This is a very important feature in order to obtain accurate unsteady computations and, by using the backward Euler scheme with a very large time step, to achieve quadratic convergence in the computation of steady state solutions (the backward Euler scheme is in fact identical to the Newton method in the limit \( \Delta t \to \infty \)).

The linear algebraic system \( Ax + b = 0 \) is solved with either the direct multifrontal method for unsymmetric matrices available in the UMFPACK package or the iterative preconditioned GMRES iterative method implemented in the SLATEC Common Mathematical Library. In the latter case both block diagonal and incomplete \( LU \) factorization preconditioners have been considered. To enhance the computational efficiency, the direct solver has been applied to the system preconditioned with the inverse of the block diagonal matrix, i.e. \( D^{-1}(Ax + b) = 0 \) has been considered. With this choice, the diagonal blocks of \( A \) reduce to the identity matrix. Notice that the two step Runge–Kutta scheme requires the solution of two systems \( Ax + b = 0 \) at each time step, but, since the matrix \( A \) is the same for the two steps, only one factorization is required.

5. Numerical results

The method presented in the previous sections has been applied to compute the steady turbulent flow over a flat plate and the unsteady turbulent vortex shedding phenomenon past the trailing edge of a turbine blade.

5.1. Turbulent boundary layer over a flat plate

The flat plate flow here considered is that reported by Wieghardt [16] and included in the 1968 AFOSR-IFP Stanford Conference [17]. This test case is one of the validation cases used by the NPARC Alliance to evaluate the accuracy of its CFD codes, in particular the turbulence models implementations and their sensitivity to near-wall grid spacing [18].

For the purpose of convergence acceleration, the Mach number ahead of the plate, equal to 0.096 in the experiment, has been increased up to 0.2 in the simulations. The Reynolds number based on the plate length is equal to \( 11.1 \times 10^6 \) and the inlet turbulence quantities are determined from the prescribed turbulence intensity and turbulent Reynolds number, both equal to \( 10^{-4} \).
For our computations we have used the same grids of the NPARC Alliance study and the results here presented are those obtained on the grids denoted by \( y_1^+ = 1 \), \( y_2^+ = 5 \) and \( y_3^+ = 30 \) in the NPARC study. The above values are representative of the maximum \( y_1^+ \) value at the grid points nearest to the plate as computed by the NPARC WIND code. The computational grids are simple Cartesian meshes and contain \( 110 \times 80 \) quadrilateral elements with 96 elements lying on the plate and 14 elements ahead of the plate leading edge. Grid lines are clustered both around the plate leading edge and near the wall surface. The distance of the first grid line parallel to the plate, divided by the plate length, is equal to \( y_1 = 6.437 \times 10^{-7} \), \( y_2 = 4.288 \times 10^{-6} \), and \( y_3 = 2.543 \times 10^{-5} \), for the three considered grids. The grids are here labelled as G1, G2 and G3 and we remark that the \( y_1^+ \) values computed by our code over almost the entire plate are much lower. In fact we have found \( y_1^+ = 0.25 \), \( y_2^+ = 1.67 \) and \( y_3^+ = 10 \) for the G1, G2 and G3 grids, respectively.

All the solutions have been computed through a sequence of P0, P1 and P2 approximations, starting from an uniform flow field at freestream conditions and using the backward Euler scheme for time integration. In all cases the P0 solution has been run for 500 time steps and after that the P1 and P2 approximations took just few tens of time steps to reach fully converged high-order solutions.

The skin friction distributions of the P1 and P2 solutions computed on the G1, G2 and G3 grids are compared with the experimental data in Figs. 1 and 2. These figures clearly show that even the P1 solution on the near-wall coarsest G3 grid is very accurate as far as the skin friction coefficient is concerned.

The near-wall behaviour of the P1 and P2 solutions is presented in terms of velocity and turbulence properties profiles. The quantities \( u^+ = u/u_\infty \), \( k^+ = k/u_\infty^2 \), and \( \omega^+ = \omega u_\infty/u_\infty^2 \) are plotted as functions of \( y^+ = yu_\infty/v_\infty \) at \( x/L = 0.923 \) (the last measuring station of the Wieghardt data). Notice that the lines of the computed results drawn in the plots depict the actual high-order polynomial approximation of the solution inside the elements.

![Friction coefficient along the plate, P1 elements.](image-url)
In the plots of the near-wall $u^+$ and $\omega^+$ profiles we have also drawn the corresponding law-of-the-wall profiles according to the equations

\[
\begin{align*}
    u^+ &= y^+ & \text{in the viscous sub-layer} \\
    u^+ &= \frac{1}{\kappa} \log y^+ + B & \text{in the log-layer,}
\end{align*}
\]

\[
\begin{align*}
    \omega^+ &= \frac{\omega_w^+}{\sqrt{\frac{2}{\beta^+} y^+ + 1}} & \text{in the viscous sub-layer} \\
    \omega^+ &= \frac{1}{\sqrt{\beta^+} y^+} & \text{in the log-layer,}
\end{align*}
\]

where $\kappa = 0.41$ is the von Karman constant, $B = 5$ and $\beta^+$ and $\beta$ are model constants. Observe that, as $\omega_w$ depends on the grid spacing, there are three curves representing the law-of-the-wall in the viscous sub-layer.

Figs. 3 and 4 show that all the computed velocity profiles agree pretty well with each other, with the exception of the P1, G3 and, to a lesser extent, of the P2, G3 solutions in the viscous sub-layer. The turbulent kinetic energy profiles, reported in Figs. 5 and 6, display plateau and peak values of $k^+$ which are in fairly good agreement with the “average” experimental data reported in [19]. Notice however that on the G3 grid even the P2 solution is not accurate enough to eliminate the inaccuracy of the P1 solution in the region of maximum $k^+$. Finally, Figs. 7 and 8 put in evidence the influence of the near-wall grid resolution and of the degree of polynomial approximation on the accuracy of the $\omega^+$ solution in the viscous sub-layer.

5.2. Unsteady vortex shedding behind a turbine blade

In the second test case we consider the subsonic flow in a cascade of turbine blades with rounded and thick trailing edges (TE). The experimental investigation of this cascade has been the
topic of a European research project, see [20,21], aimed at studying the unsteady wakes behind the trailing edge of turbine blades. The experimental results have shown that the dominant vortex shedding frequency does not appear as a single peak in the frequency spectrum but rather as a bandwidth. The frequency range is strongly influenced by the thickness and by the type (laminar, turbulent or transitional) of the boundary layers that develop on the blade pressure (PS) and suction (SS) sides.
According to the experiments, see [20], this test case has been computed for a downstream isentropic Mach number $M_{2is} = 0.4$, a Reynolds number based on the downstream isentropic conditions and on the blade chord $Re_{C2is} = 2 \times 10^6$ and an inlet turbulence intensity $Tu_{in} = 1.15 \times 10^{-2}$. The computational grid, shown in Figs. 9 and 10, contains 5411 triangular elements and consists of a layer of anisotropic elements all around the blade surface connected with the set of isotropic elements that fill the rest of the computational domain.

Fig. 5. Non-dimensional turbulent kinetic energy profile, P1 elements.

Fig. 6. Non-dimensional turbulent kinetic energy profile, P2 elements.
The solution has been computed by using P0, P1 and P2 elements. Starting from an initial flow field at rest, the P0 solution is advanced in time for 700 time steps by means of the backward Euler scheme. The time step magnitude $\Delta t$ in this phase corresponds to about 1/20 of the vortex shedding period $T$. Starting from the computed P0 flow field the integration in time continues for 800 time steps using the P1 approximation, the second-order Runge–Kutta scheme and a time step $\Delta t \approx T/80$. Then the solution switches to P2 approximation and continues, using the same integration scheme and the same time step of the P1 solution, for about 100 vortex shedding
periods. For comparison purposes also the P1 computation has been continued for the same number of vortex shedding cycles of the P2 solution.

Figs. 11 and 12 show a snapshot of the Mach number and of the turbulence intensity fields of the P2 solution. In these figures the alternate vortex shedding structure and the marked widening of the turbulent wake are clearly evident. Fig. 13 presents the evolution of the density field during
a vortex shedding cycle. The vortex shed by the pressure side appears to be stronger than that shed by the suction side. This is due to the different boundary layer development on the two sides of the blade.

Next we focus on the pressure field around the TE, where the origin of the curvilinear coordinate $s$, pointing towards the PS, is fixed on the center of the TE. The base pressure distribution
is illustrated by Figs. 14 and 15, which show the time averaged pressure distribution, made non-dimensional by the inlet total pressure, and the RMS of the pressure fluctuations, made non-dimensional by the downstream dynamic pressure. The averaging time interval corresponds to the last 28 vortex shedding periods out of the more than 100 that have been computed. The experimental data, reported in [20], are not indicated in the figures. However, we can say that, whereas the computed average pressure compares favourably with the experimental data, the RMS of the computed pressure fluctuations is largely overestimated, especially for the P2 approximation, with respect to the experiments. This discrepancy has also been observed by other authors, see [22,23], who have used different numerical methods and
turbulence models. However it is worth mentioning that subsequent experimental investigations on the same blade at an higher Mach number resulted in much greater pressure fluctuations.

The pressure fluctuations at the PS and SS separation points are displayed in Figs. 16 and 17 for both P1 and P2 solutions. Figs. 18 and 19 show the results of the FFT analysis of the pressure fluctuations.
fluctuations shown in Figs. 16 and 17. Maximum peaks occur in the range of Strouhal numbers \( S = fD_{TE}/v_2 \) between 0.26 and 0.29. This result is in excellent agreement with the experimental data reported in [20,21].

Despite the large number of computed vortex shedding cycles, Figs. 16 and 17 indicate that the flow regime is not periodic. This is in disagreement with the computational results reported.
in [22,23] where the authors claim that flow periodicity is obtained after a number of shedding cycles lower than the one of the present work. This discrepancy could be explained by considering that in [22,23] turbulence effects are accounted for by means of algebraic models, whereas in the present work we use the $k-\omega$ model which can simulate the transition in the boundary
We have noticed, in fact, that the pressure fluctuations in the trailing edge region propagate upstream and influence the location of the boundary layer transition.

However, the most striking feature of these figures is the fact that pressure fluctuations appear to be modulated in amplitude. This phenomenon surely deserves a deeper investigation in order to clarify both physical and numerical aspects that could have determined it.

Finally, the residuals time histories shown in Figs. 20 and 21 confirm that a perfectly periodic flow regime has not been reached yet.
6. Concluding remarks

We have here presented the application of a DG space discretization and an implicit Runge–Kutta time integration method to the numerical solution of the RANS and $k-\omega$ turbulence model equations. Detailed description has been given of a robust implementation of the $k-\omega$ turbulence model and of the DG space discretization scheme employed, with particular emphasis on the treatment of the viscous terms of the RANS and $k-\omega$ turbulence model equations.

The method has been validated by considering two test cases, the simple but well documented flow over a flat plate and the more complex unsteady flow which develops behind a turbine blade. All the tests have been performed by using both piecewise linear and quadratic elements (corresponding to second- and third-order accurate space discretizations). The backward Euler scheme has been used for the steady computations whereas the second-order accurate implicit Runge–Kutta scheme has been employed in the unsteady test case.

The comparison of the computed results with those available in the literature (either numerical or experimental) shows the effectiveness of the proposed approach. Notice in particular the accuracy of the results obtained with the P2 computations, which compares very well with the state-of-the-art despite the much coarser grid here employed. This is a remarkable result which is a direct consequence of the combination of the accuracy allowed by DG space discretization methods and of the effective implementation of the $k-\omega$ turbulence model as outlined in the paper.

The issue of computational efficiency has not been addressed in the present paper despite its obvious practical relevance, and a comparison between the computational resources required to reach a prescribed level of accuracy by the proposed solution approach and by more standard numerical techniques should certainly be performed.

References


