1. Examine a gas of non-interacting spin 1 particles with energy \( H = \frac{p^2}{2m} - \mu_0 s_z B \)

Solution: The occupation number of each state is

\[
n_s = e^{\beta p^2 / 2m - \mu_0 s_z B - \beta \mu - 1}
\]

The total number of particles with each spin is

\[
N_s = \frac{V}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta p^2 / 2m - \mu_0 s_z B - \beta \mu - 1}} = \frac{V}{\lambda^3} f_{3/2}(ze^{\mu_0 B})
\]

where \( f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx x^{m-1}}{e^x - 1} \), \( \lambda = \hbar / \sqrt{2\pi mk_B T} \) and \( z = e^{\beta \mu} \)

The magnetization is

\[
M = \mu_0 (N_+ - N_-) = \mu_0 \frac{V}{\lambda^3} [f_{3/2}(ze^{\mu_0 B}) - f_{3/2}(ze^{-\mu_0 B})]
\]

For small magnetic field we have the expansion

\[
f_{3/2}(ze^{\mu_0 B}) \approx f_{3/2}(1 + s\mu_0 B) + sz\mu_0 B \frac{\partial}{\partial z} f_{3/2}(z)
\]

So

\[
M = \frac{2\mu_0^2 V}{k_B T \lambda^3} B f_{1/2}(z)
\]

and

\[
\chi = (\frac{\partial M}{\partial B})_{B=0} = \frac{2\mu_0^2 V}{k_B T \lambda^3} B f_{1/2}(z)
\]

Bose-Einstein condensation happens when

\[
n = \frac{3}{\lambda^3} f_{3/2}(1)
\]

which leads at a temperature

\[
T_c(n) = \frac{\hbar^2}{2\pi m k_B} \left( \frac{n}{3\zeta_{3/2}} \right)^{2/3}
\]

where \( \zeta_{3/2} = f_{3/2}(1) \approx 2.61 \) and the susceptibility diverges.

2. Calculate the critical temperature of a non-interacting non-relativistic boson gas in a cubic box of length \( L \)

Solution:

\[
N - N_0 \approx \int_0^\infty ndN
\]

The non relativistic energy for a particle in a box is

\[
E_n = \frac{\hbar^2 \pi^2}{2mL^2} (j^2 + k^2 + l^2) = E(j^2 + k^2 + l^2)
\]
The differential number of states is
\[ dN = \frac{L^3}{(2\pi\hbar)^3} |p|^2 dp d\Omega = \frac{L^3}{(2\pi\hbar)^3} \sqrt{2m^3 E} dE d\Omega = \frac{\sqrt{E}}{16(L^3/(2\pi\hbar)^3)^{3/2}} dE d\Omega \]

The number of excited states is
\[ N - N_0 = \int n dN = \frac{4\pi}{16} \int_0^\infty dE \frac{\sqrt{E}}{\left(\frac{\pi^2 \hbar^2}{2mE}\right)^{3/2}} e^{\beta(E - \mu)} - 1 \]

Close to the transition
\[ N - N_0 = (\frac{\pi kT}{4E})^{3/2} \zeta(3/2) \]

Finally the number of particles in the ground state are
\[ N_0 = N[1 - T^{3/2} \left(\frac{4E}{\pi k}\right)^{-3/2} \left(\frac{\zeta^{3/2}(3/2)}{N^{2/3}}\right)^{3/2}] = N[1 - (T/T_c)^{3/2}] \]

3. Examine a non interacting boson gas in a three dimensional harmonic potential

**Solution:**

The energy of each state is
\[ E_n = \hbar \omega (n_x + n_y + n_z) = \hbar \omega q \]

where we have neglected the zero point energy.

\[ N - N_0 = \sum \frac{0.5(q + 1)(q + 2)}{e^{3(\hbar \omega - \mu)} - 1} \approx \int_0^\infty \frac{0.5(q^2 + 3q) dq}{e^{3(\hbar \omega - \mu)} - 1} \]

The occupation of the ground state is to lowest order
\[ N_0 = N[1 - (T/T_c)^3] \]

where
\[ T_c = \frac{\hbar \omega}{k} \left(\frac{N}{\zeta(3)}\right)^{1/3} \]