A simplified model of diffusion consists of a one-dimensional lattice, with lattice spacing $a$, in which an "impurity" makes a random walk from one lattice site to an adjacent one, making jumps at time intervals $\tau$.

1. After $N$ jumps have been made, find the probability that the atom has moved a distance $d$ from its starting point, in the limit of large $N$.

2. The diffusion coefficient is defined by the differential equation $D \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$, where $f$ is the concentration of the impurity. Find an expression for $D$ in the model described above.

Solution:

1. Suppose that the particle made $N_+$ steps forward and $N_-$ steps backward. The distance traveled is $d = a(N_+ - N_-) = as$. We have the relations $N_+ + N_- = n$ and $N_+ - N_- = s$. By solving these we get $N_+ = (n + s)/2$ and $N_- = (n - s)/2$.

The probability of traveling a distance $d$ is the probability of making $N_+$ jumps to the right, so

$$P(d) = \frac{N!}{N_+!N_-!} = \frac{N!}{(N + s)/2!(N - s)/2!}$$

Note that this is un-normalized. We will normalize it when we go to the continuum limit.

Using Stirling’s approximation we get

$$\ln P(d) = N \ln 2 - \frac{1}{2} (N + s) \ln (1 + s/N) - \frac{1}{2} (N - s) \ln (1 - s/N)$$

$P(d)$ will be sharply peaked around $s = 0$, so we Taylor expand around $s/N = 0$ and get

$$P(d) = 2^N e^{-s^2/2N}$$

By setting $x = as$ and normalizing we get

$$P(x) = \frac{1}{a \sqrt{2\pi N}} e^{-x^2/2Na^2}$$

2. The number of jumps is the time over the time it takes to make one jump, $N = t/\tau$. So

$$P(x, t) = \sqrt{\frac{\tau}{2\pi a^2 t}} e^{-x^2/2a^2t}$$

The density of concentration of impurities in the lattice is $f(x, t)$. If we start with $M$ impurities at time $t = 0$ in positions $x_1^0, x_2^0, ..., x_M^0$ then $f(x, t) = \sum_{i=1}^{M} P(x - x_i^0, t)$.

From linearity substitute $P(x)$ into

$$D \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$$

and get

$$D = a^2/2\tau$$
3. Compute the ratio of thermal conductivity of helium gas at $P = 0.1\, atm$ and $300K$ to that at $P = 0.5\, atm$ and $300K$. Compute also the ratio of viscosities at the two pressures

**Solution:** The rate of energy transfer through a unit area in the plane $Z = \text{const}$ is proportional to

$$(nv)(-\lambda dE/dT) = (nv\lambda dE/dT)(-dT/dZ)$$

The first factor is the flux of molecules in either direction across the plane. The second term is the difference in average molecular energy in a distance of one mean free path ($\lambda$). The coefficient of $(-dT/dZ)$ is the heat conductivity, $K \propto n v \lambda C$, where $C$ is the molecular specific heat.

The mean free path is $\lambda = 1/n\sigma$, where $\sigma$ is the collision cross section. In addition, $C = 3k/2$ for a monoatomic gas and $v = \sqrt{3kT/M}$. Finally

$$K \propto \sqrt{T}$$

independent of pressure, so the desired ratio is 1.

The viscosity may be calculated by considering the net transverse momentum transfer across a unit area of a plane $Z = \text{const}$, in unit time. In the same notation, this is proportional to $-(nv)(m\lambda dU/dZ)$, where $m$ is the molecular mass and $u$ is the transverse velocity. By definition, the coefficient of $du/dZ$ is the viscosity $\eta$. Thus

$$\eta \propto \sqrt{T}$$

independent of pressure, so the ratio is 1. Furthermore

$$\eta/K = \text{const}$$

4. In a hot plasma, all the atoms may be regarded as completely ionized. Although the ions have long-range forces due to Coulomb interactions, macroscopically the plasma is electrically neutral. This suggests that the Coulomb interactions are screened, and so become short-range. Estimate this range, making suitable approximations.

**Solution:**

The electrical potential in the vicinity of an ion is $\phi(r)$. The density near the ion is

$$n(r) = ne^{-e\phi(r)/kT}$$

Poisson’s equation is

$$\nabla^2 \phi(r) = -4\pi e \sum_\alpha e_\alpha n_\alpha$$

For a hot plasma

$$n_\alpha(r) = n_\alpha(1 - e_\alpha \phi/kT)$$

Substituting into the Poisson equation

$$\nabla^2 \phi(r) = 4\pi \left( \sum_\alpha \frac{n_\alpha e_\alpha^2}{kT} \right) \phi = \kappa^2 \phi$$

which has a solution

$$\phi(r) = e_{\text{ion}} \frac{e^{-\kappa r}}{r}$$

$\kappa^{-1}$ is a characteristic length scale of the plasma and is called the Debye radius.