Main message

Abstractions and Abstraction mechanisms pervade several disciplines

Evidence that computing with abstractions is becoming feasible for industrial applications

New discipline of certified computations

Goal of Workshop: Foster discussions among specialists in different areas and different levels of “maturity in the field”
A few definitions

• **Abstraction:** an operation mapping a system (of equations, ODEs, PDEs, logical statements, ...) onto another – abstract- system, whereby all what is true about the abstract system is true of the original system.

• **Robustness:** does the property of a system hold when the system undergoes perturbations of finite “size”?

• **Computations:** How can computers help us in (i) manipulating abstractions, (ii) establishing properties of systems more easily, and (iii) generating abstractions automatically?
Patrick Cousot

• Some discrete dynamical systems analysis tools can apply to certain aspects of software analysis, incl. run-time errors.

• Most characteristics (eg overflow errors) cannot be detected using straight program and variables: Too many computations or computations are not even conceivable.

• Tractability can be achieved via use of abstractions.

• Tractability can be achieved via use of overbounding invariant sets.
Abstracting state spaces and subspaces

{Set of all subsets of signed integer numbers between \(-b-1\) and \(b\})

if \(x = T\) then \(x\) is any value
if \(x = +\) then \(0 \leq x \leq b\)
if \(x = 0\) then \(x = 0\)
if \(x = -\) then \(-b-1 \leq x \leq 0\)
if \(x = \bot\) then \(\emptyset\)

Rules:
\[\pm + \pm = \pm; \quad \pm + - = T\]

Effect: Go from huge state-space decomposition to finite and simple state-space decomposition
Pablo Parrilo

- Look at algebraic/semi-algebraic optimization problems:

  Show there exists no $x \in \mathbb{R}^n$ such that $P_i(x) \leq 0$

- Relax problem to finding $M_i(x)$ such that u.t.c.

  $\sum_i P_i(x) M_i(x) > 0$ for all $x$

- Applications in control systems, comp. Geometry, combinatorial optimization etc.

- (Hyper)-Generalization of LMI techniques for dynamical system analysis, relaxation mechanisms for combinatorial problems (max-cut etc)
Jaime Peraire

• Partial Differential Equations

\[ F(u, t, \partial u/\partial x, \partial u/\partial t, \partial^2 u/\partial t^2, \ldots) = 0 \]

• Usually solved by finite difference/finite element methods

• Usual convergence results as time, space discretizations go finer.

• How to generate solutions \( u \) with absolute certificates on accuracy of solution (not \( O(\text{smthg}) \))
Bayen/Tomlin

• Computations of reachable sets for dynamical systems (hybrid/ODEs)

• Link to Optimal Control problems (reachability problem equiv. to optimal control problem)

• Link to certain PDEs (HJB equations)

• Link to “other” PDEs?
Feron

Relaxations techniques!
Other

• Upon impulse from Parrilo, completed translation of Bezout’s “General theory of Algebraic Equations” (not perfect yet, but the Latex file compiles….)

Find \( x \in \mathbb{R}^n \) such that \( P_i \left( x \right) = 0 \)

• Reduce question to single polynomial in one unknown by writing sum equation
  \[
  \sum_i P_i \left( x \right) M_i \left( x \right) = 0
  \]
  With appropriately chosen polynomial multipliers \( M_i \left( x \right) \).

Bezout theorem I about maximum number of intersections of algebraic surfaces (it’s the product of the degrees of the polynomial equations).

Bezout theorem II about coprime polynomials in one variable (\( X,Y \) coprime iff exists polynomials \( U,V \) such that \( XU + YV = 1 \)).

• To be published sometime soon by PU Press.
Computer Programs as Dynamical Systems

- A computer program can be viewed as a rule for iterative modification of the operating memory.

- In more specifics, computer program models defined by the dynamical system \( S = S(X, f, X_0, X_\infty) \) are considered.

- \( X \) is the state space, \( X_0 \subset X \), and \( X_\infty \subset X \) are sets of initial and terminal states and the set-valued function \( f : X \rightarrow 2^X \) is such that \( f(x) \subset X_\infty \) for all \( x \in X_\infty \).

- To each dynamical system \( S = S(X, f, X_0, X_\infty) \) corresponds the set of all sequences \( x = (x(0), x(1), ..., x(t), ...) \) of elements of \( X \), satisfying \( x_0 \in X_0 \), \( x(t + 1) \in f(x(t)) \), \( \forall t \in \mathbb{Z}^+ \).
Software Analysis

- Verification of essential properties of $S = S(X, f, X_0, X_\infty)$.

- Finite-time termination: $\forall x(0) \in X_0, \exists T > 0$, s.t. $x(t) \in X_\infty$, $\forall t \geq T$.

- Absence of overflow: $\forall x(0) \in X_0, \forall t \geq 0$, $x(t) \in X \subset X$. 
Abstracted Model

- An Abstracted model of a computer program is a simplified but not perfectly accurate model of the original program.

- The model $S(\widehat{X}, \widehat{f}, \widehat{X}_0, \widehat{X}_\infty)$ is an abstraction of $S(X, f, X_0, X_\infty)$ if $X \subset \widehat{X}$, $\widehat{X}_\infty \subset X_\infty$, $X_0 \subset \widehat{X}_0$, $f(x) \subset \widehat{f}(x)$, $\forall x \in X$.

- Validity of certain properties such as finite-time termination and boundedness in the abstracted model $S(\widehat{X}, \widehat{f}, \widehat{X}_0, \widehat{X}_\infty)$ imply their validity in the original model $S(X, f, X_0, X_\infty)$. 
An Example:

\[ T(1, \ldots, n); \text{ array of reals} \]
\[ I \in [I_{\text{min}}, I_{\text{max}}]; \text{ uncertain integer input} \]
\[ l = l_0; \text{ Integer within } \{1, \ldots, n\} \text{ sets the initial condition} \]
\[ \text{for } k = 1 : K \]
\[ x = |a|.T(l) + b; \]
\[ l = [c.x + d.l + e.I]; \]
end

Assume that \( M \) is the overflow limit. Let \( T_{\text{max}} := \max_{1 \leq i \leq n} T(i) \) and \( T_{\text{min}} := \min_{1 \leq i \leq n} T(i) \). if \([|a|.T_{\text{min}} + b, |a|.T_{\text{max}} + b] \subset [-M, M]\), then the variable \( x \) is guaranteed to remain bounded, provided that the index \( l \), remains within the feasible set \( \{1, \ldots, n\} \).
An Abstracted model of the above program is then give by

\[ T_{\text{min}}, T_{\text{max}}: \text{real variable} \]
\[ I \in [I_{\text{min}}, I_{\text{max}}]; \text{ uncertain real input} \]
\[ x \in [\|a\|T_{\text{min}} + b, \|a\|T_{\text{max}} + b] \subset [-M, M]; \text{ uncertain real input} \]
\[ l \in [1, n]; \text{ uncertain real initial condition/input} \]
\[ \text{for } k = 1: K \]
\[ l = c.x + d.l + e.I; \]
\[ \text{end} \]

Verify that subject to the above transformation, \( l \), remains within the interval \([1, n + 1)\).

At an abstract level, a universal language is sufficient. For practical considerations such as availability of an efficient relaxation technique and compatibility with an optimization engine, specific modeling techniques come a necessity.
Models of Computer Programs

Linear systems with conditional switching:

\[ X = \{0, 1, 2, \ldots, m\} \times \mathbb{R}^n, \ X_0 = \{0\} \times \mathbb{R}^n, \ X_\infty = \{m\} \times \mathbb{R}^n. \]

\( f : X \mapsto 2^X \) is defined by matrices \( A_k, B_k, L_k, F_k, G_k, H_k, C_k, D_k \), where \( k \in \{0, 1, \ldots, m-1\} \), as well as by a function \( p : \{0, \ldots, m-1\} \mapsto \{0, \ldots, m\} \), according to the following rule:

\[
f(k, v) = \left\{ (k + 1, A_k v + B_k w + L_k) : w \in [-1, 1] \right\}
\]

when \( C_k v + D_k \leq 0 \) and \( k < m \),

\[
f(k, v) = \left\{ (p(k), F_k x + G_k w + H_k) : w \in [-1, 1] \right\}
\]

when \( C_k v + D_k > 0 \) and \( k < m \), and \( f(k, v) = \{m, v\} \) when \( k = m \).
Mixed integer/linear systems:

\[ X = \mathbb{R}^n \] is the state space. The state transition map \( f : X \mapsto 2^X \) is defined by two matrices \( F, H \) of appropriate dimensions, according to

\[ f(x) = \{ F[x; w; v; 1] : H[x; w; v; 1] = 0, \ w \in [-1, 1]^q, \ v \in \{-1, 1\}^r \} \].

Every piecewise linear map on \( X \) can be written in this format.

Trigonometric polynomial models:

The state space \( X \) is a direct product of sets of the form \( \mathbb{T}^k \) or \( \mathbb{Z}_q^k \), where \( \mathbb{Z}_q \) denotes the set of all complex numbers \( z \) such that \( z^q = 1 \). The state transition map \( f : X \mapsto 2^X \) is defined by a vector polynomial \( p \) of \( 2n + k \) complex variables, according to

\[ f(x) = \{ y \in X : p(y, x, z) = 0 \text{ for some } z \in \mathbb{T}^k \} \].
Consider a computer program modeled as a mixed integer/linear system. If there exists a constant $\theta > 1$ and a function $V(.) : \mathbb{R}^n \rightarrow \mathbb{R}$, which satisfies

1) $V(x(0)) < 0$,  2) $V(x(k+1)) < \theta V(x(k))$,  3) $V(x(k)) > \left\| \frac{x(k)}{M} \right\| - 1$

along any trajectory of the system, then the program will not overflow and will terminate in finite time.

Appropriate system invariants for the two other suggested models of computer programs have a similar structure.

Assuming a quadratic or linear form for the function $V(.)$, the search for a system invariant, reduces to solving a finite number of LMIs or an LP respectively.
Systematic Improvement of Analysis

- Increasing the depth (memory) of the model:

\[ x(k + 2) = FX(k), \quad \text{s.t.} \quad HX(k) = 0, \quad k = 1, 2, \ldots \]

- Then, search for a function \( V(.) : \mathbb{R}^n \rightarrow \mathbb{R} \), which satisfies

\[
\begin{align*}
V(x(0)) &< 0, \quad V(x(1)) < 0, \\
V(x(k + 2)) &< \theta V(x(k)) \quad V(x(k - 2)) > \left\| \frac{x(k)}{M} \right\| - 1
\end{align*}
\]

- This can be extended by increasing the memory of the model even more.

- Recursive search for invariants: i.e. search for invariants that establish the desired properties partially, then use them as new information about the system to find stronger invariants.
An Example

Consider the program:

\[
\begin{align*}
    x_1 &= 0; x_2 = 0; \\
    \text{while } x_2 &\leq 100, \\
    \quad \text{if } x_1 &\geq 0, \\
    \quad \quad x_1 &= x_1 - a; \\
    \quad \text{else} \\
    \quad x_1 &= x_1 + b; \\
    \text{end} \\
    x_2 &= x_2 + 1; \\
    \text{end}
\end{align*}
\]

This program can be modeled as a mixed integer/linear system with one binary and two slack variables:
\[ x(k + 1) = FX(k) \] subject to the additional linear constraint \( HX(k) = 0, \ k = 1, 2, \ldots \)

where \( X(k) = [x(k); w_1(k); w_2(k); v(k); 1] \) and

\[
\begin{align*}
x_0 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
F &= \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{a+b}{2} & \frac{b-a}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
H &= \begin{bmatrix} 1 & 0 & -\frac{M}{2} & 0 & -\frac{M}{2} & 0 \\ 0 & 1 & 0 & R & 0 & R - 100 \end{bmatrix}, \\
R &= \frac{M+100}{2}.
\end{align*}
\]

\( M \) is the overflow limit.

The program should be correct for all \( a \in \left(-\frac{M}{101}, M\right) \) and \( b \in \left(-\frac{M}{101}, M\right). \)
Figure 1: Shows the rather conservative results of analysis using QLF.
Figure 2: Analysis was significantly improved by increasing the memory of the model.