Recap

Last week, we talked about using Coulomb’s law to calculate the electric field. The hardest parts are choosing the source coordinates and setting up the integral. For instance, with the ring, you want to use spherical coordinates.

We talked about an infinite line, a ring with the source on the axis, and a ring with the source off the axis. For the second case, \( \vec{r}_p - \vec{r}_s = z \hat{k} - R \hat{r} \), where the second term cancels out if you go around the full ring. For the third case, we used two coordinate systems: \( \vec{r}_p = x \hat{i} + y \hat{j} + z \hat{k} \) and \( \vec{r}_s = R \hat{r} = R (\cos \theta \hat{i} + \sin \theta \hat{j}) \).

Note: a disk is just based off of a ring. Form a ring of width \( dr \), and draw an element of arc length \( r \, d\theta \). Then the area of this element is \( dr \, r \, d\theta \).

The Big Picture

If the source is fixed, then it’s external to our system, which contains just our charge. Then the force on the charge is \( \vec{F}_q = q \vec{E}_s(P) = m_q \vec{a}_q \). Solving for the field lets us solve for the acceleration, so we can precisely determine the motion of the charge.

For example, if you have two parallel plates with a field going from top to bottom, sending a charge through them causes it to accelerate downward; this looks like gravity, or projectile motion. You could add more plates to move the particle in any direction, like a CRT TV.

Another example: In the case of an electron moving around a nucleus, you can write the equations for circular motion as\( -m_e (v^2 / r) \hat{r} = -qE_s \hat{r} = -(e^2 / r^2) \hat{r} \). Solving for \( v \) gives \( v = \sqrt{e^2 / (m_e r)} \). Now, assume that angular momentum is quantized, so electrons at radius \( r_n \) have angular momentum \( L_n = n \hbar = m_e r_n v = \sqrt{e^2 r_n m_e} \). Finally, solving for the radius gives \( r_n = (n^2 \hbar^2) / (m_e e^2) \).

Thinking Deep Thoughts

Back up for a second: What does the equal sign in \( \vec{F} = m \vec{a} \) mean? Originally, scientists defined force as \( \vec{F} = m \vec{a} \), but now they accept that they can only create inductive approximations based on empirical data. As long as these approximations are accurate to within the accuracy of our measurement capabilities, we denote this approximation by equality.

Dourmashkin: “You could have a polynomial with 83 terms, and as long as you can only measure the first 82, you’re fine.”

What the equal sign says is that the magnitude and direction of two completely different vectors are equal. That is to say, they’re not the same vector, but they do have the same magnitude and direction. For instance, if a charge is traveling in circles, we know mathematically that \( \vec{a} = (v^2 / R) \hat{r} \), and we know via Coulomb’s law that \( \vec{F} = -qE_s \hat{r} \). The equal sign lets us solve for missing quantities by saying that the magnitude and direction of \( m \vec{a} \) and of \( \vec{F} \) should be equal, even though the vectors represent different quantities.

Remember, why are we interested in finding \( \vec{E} \) fields? Because then we can determine the motion of any charged particle using \( \vec{F} = q \vec{E} = m \vec{a} \). That’s what makes the equal sign so useful.

Gauss’s Law

One way to find the electric field is to first find the flux, which is \( \int_S \vec{E} \cdot d\vec{a} \). Flux is like flow, except flow is for moving objects (in a velocity field) while flux doesn’t have anything moving (for instance, an electric
field). The symbol $\oint_S$ is a surface integral over the surface $S$ with $d\mathbf{a} = \pm (da)\hat{n}$. By convention, $\hat{n}$ points out of a closed surface.

Let $V$ be the volume inside the surface $S$. If the charge distribution is given as $\rho$, then the total charge in $S$ is $\iiint_S \rho \, dV$. Gauss’s Law relates the flux and the enclosed charge, stating that $\oint_S \mathbf{E} \cdot d\mathbf{a} = 4\pi \iiint_S \rho \, dV$. (In SI units, we would have $1/\varepsilon_0$ instead of $4\pi$.)

Conceptual question: Is the $E$ field due only to the charge enclosed? To answer that, let’s work a problem: Take a charge $q$ and let $S$ be a sphere centered at the charge with radius $r$. The right-hand side of Gauss’s Law is simply $4\pi q_s$. The left-hand side is a bit more complicated. We know that $d\mathbf{a}$ and $\mathbf{E}$ point radially outward and that $|E|$ is uniform on the surface. This means that $\oint_S \mathbf{E} \cdot d\mathbf{a} = E \oint_S da = E \cdot 4\pi r^2$. (We picked a sphere because its symmetry lets us make these arguments.) Equating these two quantities using Gauss’s Law, we see that $\mathbf{E} = (q_s/r^2)\hat{r}$ radially outward.

What about a spherical surface with a charge outside the surface? The enclosed charge is zero and the flux is zero, but the $E$ field on the surface is not zero! So no, an $E$ field is not due only to the charge enclosed. Gauss’s Law merely states that the flux is due only to the charge enclosed. To see why the flux is zero in this situation, draw a cone with vertex at the charge. The field dissipates as $1/r^2$ but the cross-sectional area of the cone grows as $r^2$, so $E \, dA$ is constant. Furthermore, the flux is inward on one side and outward on the other, so these parts cancel, and the total flux is zero.

Other situations with enough symmetry for Gauss’s Law include an infinite wire and an infinite plane. For homework, find the electric field in space caused by an infinite wire and an infinite plane. Additionally, read the proof of Gauss’s Law in Purcell, and try to describe it in your own words.

**Homework**

Proof of Gauss’s Law:

- It is easy to show that the flux is proportional to the enclosed charge when the charge is at the center of a Gaussian sphere
- Using solid angles, you can show that the flux is proportional to the enclosed charge when the charge is off-center inside a Gaussian sphere
- You can also show that charges outside the Gaussian sphere contribute no flux
- By the superposition principle, Gauss’s Law is true for any number of point charges around a Gaussian sphere, so by taking a Riemann sum, it also works for continuous charge distributions