More Gauss’s Law

Going back to the cylinder from yesterday, let’s see what happens to the field if the density varies as \( \rho(r) = br \). We now have to evaluate the integral \( \iiint \rho dV \) instead of simply taking \( \rho \) times the total volume. To save time, we don’t have to integrate three variables; \( \rho \) depends on \( r \), so we only have to integrate over \( r \).

Draw a cylinder of radius \( r' \), thickness \( dr' \), and length \( l \), so the volume element is \( dV = 2\pi r' l \, dr' \). (Alternatively, we could integrate over three variables and show that \( \iiint r' \, dr' \, d\theta' \, dz' = 2\pi l \int r' \, dr' \).) When \( r < R \), the enclosed charge is \( \int_0^r (br')(2\pi r' l \, dr') = 2\pi l \int_0^r r'^2 \, dr' = (2/3)\pi l br'^3 \), so Gauss’s Law gives \( 2\pi rlE = (8/3)\pi^2 blr^3 \) and \( E = (4/3)\pi br^2 \). When \( r > R \), we change the enclosed charge integral to run from 0 to \( R \) instead.

It’s crucial to know formulae for geometry. For instance, for a spherical shell, \( dV = (dr') (4\pi r'^2) \).

What to look for when we are doing a Gauss’s Law problem:

1. Is there enough symmetry in the field to pick a Gaussian surface?
2. Is there enough symmetry in the charge density?

Example: Given a concentric sphere (radius \( a \), density \( \rho_1 = b/r \)) inside a spherical shell (inner radius \( b > a \), outer radius \( c > b \), constant density \( \rho_2 \)), find the electric field at a radius \( b < r < c \). (If we solved for all \( r \), we’d find that there’s one electric field, but it’s represented as a piecewise function with four parts; we need to solve for each part separately.) The flux is \( E \cdot 4\pi r^2 \) and the enclosed charge is \( \rho_2 (4\pi r^3/3 - 4\pi b^3/3) + \int_0^a (b/r') 4\pi r'^2 \, dr' \).

Now take a spherical shell of radius \( R \) with charge \( +Q \). Inside the shell, there is no enclosed charge, so Gauss’s Law tells us that there is no electric field. Outside the shell, the enclosed charge is \( +Q \), so \( E(4\pi r^2) = 4\pi (+Q) \) and \( E = Q/r^2 \).

This raises two important points:

1. Outside a solid sphere of charge, you can treat it like a point charge.
2. Inside a solid sphere of charge, you only have to consider the charge within that radius.

Gravity

An extension: Because gravity is also an inverse square law, we can write a “Gauss’s Law for gravity.” Let’s define the gravitational field as:

\[
\vec{g} = \frac{\vec{F}_{mi}}{m_i} = -\frac{Gm_c m_i/r^2}{m_i} \hat{r}_{ct} = -\frac{Gm_c \hat{r}_{ct}}{r^2}
\]

Thus, \( \int \vec{g} \cdot d\vec{a} = -4\pi Gm_{ene} \). At a distance \( r \) away, this gives \( g4\pi r^2 = -4\pi Gm_c \), so \( g = -Gm_c/r^2 \). This easily shows you can treat a planet as a point mass—a fact that took Newton a very long time to prove!
One problem involving gravity is the “annoying roommate problem.” You have an annoying roommate, so you drop them into a slit in the Earth. Using Gauss’s Law for gravity, you can show:

\[ \vec{F} = m\vec{a} \]
\[ -\frac{G m_{\text{enc}} m_p}{x^2} = m_p \frac{d^2 x}{dt^2} \]
\[ -\frac{G \rho \frac{4}{3} \pi x^3}{x^2} = \frac{d^2 x}{dt^2} \]

This looks just like \( \frac{d^2 x}{dt^2} = -k x/m \), which has frequency \( \omega_0 = \sqrt{k/m} \). This differential equation is similar, so we can see that the roommate has frequency \( \omega_0 = \sqrt{G \rho (4/3) \pi} = 2\pi/T \).

Coming up next class: an infinite sheet, boundary conditions, pressure on a surface charge, and parallel plates. Energy will be on Monday.