Thoughts About Gauss’s Law

Take an infinite sheet of uniform charge density \( \sigma \). Draw as a Gaussian surface a cylinder. Then \( 2EA_{\text{cap}} = 4\pi \sigma A_{\text{cap}} \), so \( |E| = 2\pi \sigma \).

Now take two parallel infinite plates, the top having charge density \(+\sigma\) and the bottom having charge density \(-\sigma\). The superposition principle tells us that the field above the top plate is \( 2\pi \sigma - 2\pi \sigma = 0 \). Similarly, the field below the bottom plate is 0. Between the plates, the field is \( 2\pi \sigma + 2\pi \sigma = 4\pi \sigma \). In other words:

\[
\vec{E} = \begin{cases} 
0 & z > d \\
-4\pi \sigma \hat{k} & 0 < z < d \\
0 & z < 0 
\end{cases}
\]

Notice that, if we argue (via symmetry) that the field is zero outside of the plates, then drawing a Gaussian cylinder with one cap outside the plates and one cap between the plates shows that \( EA_{\text{cap}} = 4\pi \sigma A_{\text{cap}} \), which is another way to show that \( E = 4\pi \sigma \). The only charge contributing to the flux is the charge enclosed, but all the charge is contributing to the field.

Do not make the mistake of saying: “Only one plate is enclosed, so let’s double our result and say the field is \( 8\pi \sigma \).” You’ve already accounted for the other plate, even though it’s not enclosed! (Sometimes it is true that only the enclosed charge is contributing to the field, such as when you draw two concentric spheres and draw a Gaussian sphere between them. But you cannot conclude this in general, as shown by the parallel plate example.)

Another example: Inside a spherical shell, Gauss’s Law tells us that the flux through any Gaussian sphere is zero. We can argue by symmetry that the field is uniform, so \( 4\pi r^2 \) and the field is zero inside the shell. But again, this isn’t true in general! If we remove a small spherical cavity from a uniform spherical charge distribution, the flux through any Gaussian sphere inside the cavity is still zero, but the field is not zero. It may look at first like we can’t find the field but we can, using the law of superposition.

(Tangent: Superposition works because \( \vec{E} = \hat{F}/q \), and force is a vectorial quantity. It’s a tremendous observation that the two macroscopic forces, electromagnetism and gravity, are vectorial quantities. However, some interactions are non-vectorial, such as spin interactions. Superposition does not hold for these interactions.)

Inside the cavity, let \( \vec{r} \) be the vector from the center of the large sphere, and \( \vec{r}' \) be the vector from the center of the cavity. Using superposition, we can write \( \vec{E}(P) = \vec{E}_{+\rho} + \vec{E}_{-\rho} = \rho_4 \pi (\vec{r} - \vec{r}') \). But inside the cavity, \( \vec{r} - \vec{r}' \) equals the vector connecting the two centers, so it’s constant. This means that the field is constant inside the hole! (The reason this breaks down outside the hole is that \( E_{-\rho} = -\rho_4 \pi r' \hat{r} \) no longer holds, since the enclosed charge is a constant. Now, \( \vec{E}(P) = \rho_4 \pi [\vec{r} - (b^3/r' r^2) \hat{r}'] \), which is angle-dependent.)

Force on a Charged Surface

Changing topics, let’s look at a spherical shell of charge \( Q \), radius \( R \), and charge density \( \sigma \). It’s easy to show that the field is \( \vec{E} = 0 \) just inside the shell and \( \vec{E} = 4\pi \sigma \hat{r} \) just outside the shell. Now, let’s partition the sphere into a small disc of charge \( dq = \sigma \, da \) and the rest of the sphere. Thus, \( \vec{E}_{\text{shell}} = \vec{E}_{\text{else}} + \vec{E}_{\text{disc}} \). Then the force on the disc is \( dF = \vec{E}_{\text{else}}dq \), since the disc can’t exert a force on itself.

At a point \( P \) just outside the disc, the field is \( \vec{E}_{\text{else}} = \vec{E}_{\text{sphere}} - \vec{E}_{\text{disc}} = 4\pi \sigma \hat{r} - 2\pi \sigma \hat{r} = 2\pi \sigma \hat{r} \). Thus, the force on this small disc is \( dF = 2\pi \sigma \hat{r} dq \). This is the same result as in the book, but derived in a different way. (Intuitively, the part of the \( \vec{E} \) field that contributes to the force should be less than the \( \vec{E} \) field itself because we have to subtract out the \( \vec{E} \) field caused by the disc itself. This field is substantially large since we’re so close to the source. In this case, subtracting out the self-field cuts our field in half.)
Looking at the question on the problem set, we see that we just want to consider the $z$-component of the force, $dF_z = dF \cos \theta = 2\pi \sigma \cos \theta \ dq$. Using spherical coordinates, $dq = \sigma \ da = \sigma R^2 \sin \theta \ d\theta \ d\phi$. So, $dF_z = 2\pi \sigma^2 R^2 \sin \theta \ \cos \theta \ dq$. Integrating (with steps skipped), $F_z = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} dF_z = 2\pi \sigma^2 R^2 = Q^2/(8R^2)$.

When you finish a pset, go back and think about the big idea of each problem. What did you learn from doing it? And when the solutions come out, read them! Don’t just think of the pset as a job to complete.

Dourmashkin: “Every problem we solve is like a brick in the edifice of our knowledge.”

Up next, we’ll be talking about potential energy, including three ways to calculate it. Additionally, we need to understand div, grad, and curl, both mathematically and conceptually. We’ll also need to understand the concepts behind the proofs for Stokes’ Theorem, the Divergence Theorem, and the Fundamental Theorem of Vector Calculus.

**Work-Kinetic Energy Theorem**

If we apply an external force $F_{\text{ext}}$ to a mass $m$, bringing it from point $A$ to point $B$, then $W_{\text{object}} = \int_A^B \vec{F}_{\text{ext}} \cdot d\vec{s} + \int_A^B \vec{F}_{\text{grav}} \cdot d\vec{s}$. But if $KE(A) = KE(B) = 0$, then the total work is zero because $\Delta KE = \int_A^B \vec{F}_{\text{total}} \cdot d\vec{s}$.

The forces may not be the same (and, in fact, $\vec{F}_{\text{ext}} > \vec{F}_{\text{grav}}$ in order to move the mass upwards), but the overall values for work are equal and opposite so that they sum to zero.

Where does this come from? Mathematically,

$$\int_{x_0}^{x_f} a_x dx = \int \frac{dv_x}{dt} v_x dt = \int v_x dv_x = \left. \frac{1}{2} v_x^2 \right|_{v_0}^{v_f} = \frac{1}{2} (v_f^2 - v_0^2).$$

Performing this calculation for all three components and summing each one times $m$, we see that $\Delta KE = \frac{1}{2} m(v_{xf}^2 + v_{yf}^2 + v_{zf}^2) - v_{0x}^2(0) - v_{0y}^2(0) - v_{0z}^2(0) = \int (ma_x dx + ma_y dy + ma_z dz) = \int_0^f \vec{F}_{\text{total}} \cdot d\vec{s}$. In other words, we used Newton’s law $\vec{F}_{\text{total}} = m\vec{a}$ to derive the work-kinetic energy theorem $\Delta KE = \int_0^f \vec{F}_{\text{total}} \cdot d\vec{s}$.

We’ll talk more about this on Monday. In the meantime, we should read Chapter 2.