13 Geometric Series, Power Series, Ratio Test

Geometric Series
A geometric series has the form

\[ 1 + r + r^2 + r^3 + \ldots = \sum_{n=0}^{\infty} r^n \]

Or more generally

\[ a + ar + ar^2 + ar^3 + \ldots = \sum_{n=0}^{\infty} ar^n \]

We call the number \( r \) the ratio of the geometric series. That is, you multiply each term by \( r \) to find the next term in the series.

**Punch line.** The geometric series

\[ 1 + r + r^2 + r^3 + \ldots \begin{cases} \text{converges to} & \frac{1}{1-r} \quad \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases} \]

**Example 13.1.** Find \( \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \).

**answer:** Writing the sum out we see it is a geometric series with ratio \( r = 1/2 \).

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1-1/2} = 2 \]

**Example 13.2.** Compute \( \sum_{n=0}^{\infty} (-\frac{1}{3})^n \).

**answer:** This is a geometric series with ratio \( 1/3 \).

\[ 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots = \frac{1}{1-(-1/3)} = \frac{3}{4} \]

**Example 13.3.** Does \( \sum_{n=0}^{\infty} \frac{1}{(1+(n-1)/n)^n} \) converge?

**answer:** This is not a geometric series, but we can compare it asymptotically with the geometric series \( \sum 2^{-n} \). Since this converges, so does the series in question.

(If we wanted to we could compare the series directly to \( \sum 1.5^{-n} \).)

Variations
\[
\sum_{n=5}^{\infty} \left( \frac{3}{4} \right)^n = \left( \frac{3}{4} \right)^5 + \left( \frac{3}{4} \right)^6 + \left( \frac{3}{4} \right)^7 + \ldots = \left( \frac{3}{4} \right)^5 \left( 1 + \frac{3}{4} + \left( \frac{3}{4} \right)^2 + \ldots \right)
\]

\[
= \left( \frac{3}{4} \right)^5 \left( \frac{1}{1 - 3/4} \right) = \left( \frac{3}{4} \right)^5 \cdot 4.
\]

\[
\sum_{n=2}^{\infty} \pi r^n = \pi r^2 (1 + r + r^2 \ldots) = \pi r^2 \left( \frac{1}{1 - r} \right) \text{ (provided } |r| < 1).\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^{2n} = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right)^n = \frac{4}{3}.
\]

**Proof** (of the sum of geometric series)

Using the \(N^{th}\) partial sum we have

\[
S_N = 1 + r + r^2 + \ldots + r^N
\]

\[
rS_N = r + r^2 + r^3 + \ldots + r^{N+1}
\]

Subtract the second equation from the first to get

\[
(1 - r)S_N = 1 - r^{N+1}
\]

Thus, \(S_N = \frac{1 - r^{N+1}}{1 - r} \).

Taking limits: \(\lim_{N \to \infty} S_N = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } r \geq 1 \\ \text{doesn't exist} & \text{if } r \leq -1 \end{cases} \)

**Example 13.4.** Find a closed form for \(f(x) = x + x^3/3 + x^5/5 + \ldots \)

**Answer:** Taking derivatives: \(f'(x) = 1 + x^2 + x^4 + x^6 + \ldots = 1/(1 - x^2)\)

Integrating: \(f(x) = \int \frac{1}{1 - x^2} \, dx \).

Using partial fractions) \(\frac{1}{1-x^2} = \frac{1/2}{1-x} + \frac{1/2}{1+x} \).

Thus \(f(x) = -\frac{1}{2} \ln(|1 - x|) + \frac{1}{2} \ln(|1 + x|) = \ln \sqrt{\frac{|1 + x|}{|1 - x|}} + C.\)

We find the value of \(C\) by letting \(x = 0\): \(f(0) = 0 \Rightarrow C = 0\). Thus,

\[
f(x) = \ln \sqrt{\frac{|1 + x|}{|1 - x|}}.
\]
**Ratio Test.** Consider the series $\sum a_n$. Assume: $a_n > 0$ for all $n$.

Define $L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$.

If $L < 1$ then $\sum a_n$ converges.
If $L > 1$ then $\sum a_n$ diverges.
If $L = 1$ then the test fails.
If there is no limit $L$ the the test fails.

**Example 13.5.** Does $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ converge or diverge?

**answer:** The terms are all positive so we can try the ratio test:

$$L = \lim_{n \to \infty} \frac{2^{n+1}/(n+1)^3}{2^n/n^3} = \lim_{n \to \infty} 2 \frac{n^3}{(n+1)^3} = 2$$

Since $L > 1$, the series diverges.

**Example 13.6.** Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge or diverge?

**answer:** The terms are all positive so we can try the ratio test:

$$L = \lim_{n \to \infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n \to \infty} 2/(n+1) = 0.$$ 

Since $L < 1$, the series converges.

**Example 13.7.** Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

**answer:** The terms are all positive so we can try the ratio test:

$$L = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1.$$ 

Since $L = 1$ the ratio test fails.

In this case, we know the series converges using the $p$-test or the integral test.

**Absolute convergence.** Series of all positive terms are better behaved than other series. As you add in more terms the series keeps growing. Either the sum reaches a limit or else it grows to infinity.

We can make any series one with positive terms by taking absolute values. This leads to the definition of absolute convergence.

Given a series $\sum a_n$, if the ‘absolute series’ $\sum_{n=0}^{\infty} |a_n|$ converges then we say that the series $\sum a_n$ converges absolutely.

If a series converges but does not converge absolutely then we say it is conditionally convergent.
Theorem. If the series $\sum_{n=0}^{\infty} a_n$ converges absolutely then the series $\sum a_n$ converges in the usual sense.

Power Series Convergence.

Given a power (Taylor) series $\sum_{n=0}^{\infty} a_n x^n$ we are often interested in knowing for what values of $x$ the series converges. We can often use the following theorem to simplify matters.

Theorem. If $\sum_{n=0}^{\infty} |a_n x^n|$ converges then so does $\sum_{n=0}^{\infty} a_n x^n$.

Proof. This is just the previous theorem which says that if the series converges absolutely, then it converges in the usual sense.

Ratio Test examples for power series:

Example 13.8. Find all values of $x$ for which $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges absolutely.

**Answer:** Taking the absolute ratio of successive terms: $\frac{|x^{n+1}/(n+1)|}{|x^n/n|} = |x| \cdot \frac{n}{n+1}$.

Taking the limit: $L = \lim_{n \to \infty} |x| \cdot \frac{n}{n+1} = |x|$.

By the ratio test we know the series converges for $|x| < 1$ and diverges for $|x| > 1$.

(For $|x| = 1$ we need to look more carefully.)

Example 13.9. Find all values of $x$ for which $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges absolutely.

**Answer:** Taking the ratio we have $\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = |x| \cdot \frac{1}{n+1}$.

Taking the limit: $L = \lim_{n \to \infty} |x| \cdot \frac{1}{n+1} = 0$.

Since this is less than 1 for all $x$, the series converges absolutely for all $x$.

Example 13.10. For which $x$ does $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2^n n}$ converge absolutely?

**Answer:** The ratio is $\frac{|x^{2n+3}/(2^{n+1}(n+1))|}{|x^{2n+1}/(2^n n)|} = |x|^2 \cdot \frac{n}{2(n+1)}$.

Taking the limit: $L = \lim_{n \to \infty} |x|^2 \cdot \frac{n}{2(n+1)} = \frac{x^2}{2}$.

So, the series converges absolutely when $\frac{x^2}{2} < 1$, i.e. for $|x| < \sqrt{2}$.

Alternating series test.

If $a_n$ alternates sign and $|a_n| \downarrow 0$ then $\sum a_n$ converges.

Example 13.11. (Alternating harmonic series) Show that the alternating harmonic series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \] converges conditionally.

**answer:** Clearly the terms alternate in sign and go to 0, so the series converges. To show it converges conditionally, we have to show that it does not converge absolutely. The absolute series is
\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]
This is the harmonic series, which we know diverges. QED.

**Note.** It is possible to show the alternating harmonic series converges to \( \ln(2)! \) (See the text by Simmons: bottom page 457 through example 3.)