23  Kepler’s second law

What I have here is just a brief version of the notes §K. I suggest you read that instead.

Claim: If a body moves under a central force then it sweeps out equal areas in equal time.

Proof: Note a central force means $\vec{r}$ is parallel to $\vec{a}$.

In a short time $\Delta t$ the position vector sweeps out an area $\Delta A$.

Using vectors we see $\Delta A \approx \frac{1}{2} |\vec{r} \times \Delta \vec{r}|$. So, $\frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \frac{d\vec{r}}{dt}|$.

Equal areas in equal time means $\frac{dA}{dt} = \text{constant}$.

Consider $\vec{w} = \vec{r} \times \frac{d\vec{r}}{dt}$. By the product rule

$$\frac{d\vec{w}}{dt} = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

$$= \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \vec{a}$$

Both terms are 0 since $\vec{a}$ is parallel to $\vec{r}$. Thus

$$\frac{d\vec{w}}{dt} = 0 \Rightarrow \frac{dA}{dt} = \text{constant}.$$