24 Functions of several variables, partial derivatives

Example 24.1. Functions of several variables

\[ f(x, y) = x^2 + y^2 \Rightarrow f(1, 2) = 5 \text{ etc.} \]

\[ f(x, y) = xy^2 e^{x+y} \]

\[ f(x, y, z) = xy \log z \]

Ideal gas law: \( P = kT/V \)

24.1 Dependent and independent variables

In \( z = f(x, y) \) we say \( x, y \) are independent and \( z \) is dependent.

This indicates that \( x \) and \( y \) are free to take values and then \( z \) depends on these values.

For now it will be clear which are which, later we’ll have to take more care.

24.2 Partial derivatives

Definition. The partial derivative of \( f \) with respect to \( x \) is denoted \( \frac{\partial f}{\partial x} \). It is computed by differentiating with respect to \( x \) holding all other independent variables fixed –i.e. pretend they are constant.

Example 24.2. Suppose \( f(x, y) = x^2 y + y^2 + x^2 - 3 \). Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

answer: \( \frac{\partial f}{\partial x} = 2xy + 2x, \quad \frac{\partial f}{\partial y} = x^2 + 2y. \)

24.2.1 Notation

Suppose that \( z = f(x, y) \). Then we use all of the following notations for \( \frac{\partial f}{\partial x} \):

\[ \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x. \]

If we want to evaluate it at a point we write

\[ \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = f_x(x_0, y_0) = \left. \frac{\partial f}{\partial x} \right|_0 = z_x(x_0, y_0). \]

Example 24.3. Suppose \( f(x, y) = x^2 y + y^2 + x^2 - 3 \). Compute the partial with respect to \( x \) and evaluate it at the point \((1, 2)\).

answer: \( \frac{\partial f}{\partial x} = 2xy + 2x, \quad \left. \frac{\partial f}{\partial x} \right|_{(1,2)} = 2 \cdot 1 \cdot 2 + 2 \cdot 1 = 6. \)
24.2.2 Higher order partials

Example 24.4. Let \( f(x, y) = x^2 y^3 + xy + x^3 + y^4 + 2 \). Compute \( \frac{\partial^2 f}{\partial x \partial y} \).

**Answer:** First compute \( \frac{\partial f}{\partial x} = 2xy^3 + y + 3x^2 \). Then compute the partial with respect to \( y \) of \( \frac{\partial f}{\partial x} \):

\[
\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 + 1.
\]

Note: we can compute the derivatives in either order and get the same answer, i.e.

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2 + 1.
\]