25 Tangent plane, level curves, contour surfaces

25.1 Graphs, contour lines and level curves

Suppose $y = f(x)$. This has 1 independent variable and 1 dependent variable, so we need 2 dimensions to graph it.

In painful detail the technique we use to make the graph is:

Go to $x$; then compute $y = f(x)$; then go up to height $y$ and plot a point.

To plot $z = f(x, y)$ we do essentially the same.

Example 25.1. Graph the function $z = f(x, y) = a^2 x^2 + y^2$

**answer:** Go to $(x, y)$
Compute $z = f(x, y)$
Go up to height $z$ and plot the point.

**Definition.** If we slice the graph with a horizontal plane the intersection is a **contour curve** of the function. The projection of the contour curve onto the $xy$-plane is called a **level curve** of the function.

Example 25.2. Find the contour and level curves of $z = 4x^2 + y^2$ at height $z = 3$

**answer:** Here are the steps:
Slice the graph (surface) at height $z = 3$. This is a contour curve. It consists of the points $(x, y, 3)$ where $z = 3 = 4x^2 + y^2$.

The level curve is in the $xy$-plane. It is simply the points $(x, y)$, where $3 = 4x^2 + y^2$. That is, the level curve is an ellipse in the $xy$-plane.

The idea is the same as for topographic maps.
To repeat:
Contours are the curves \textit{on the graph} at a given height. They sit above the level curves.
The terminology here is not always followed. Some people use level curves and contours interchangeably. You’ll have to rely on context to know if the curves are in the $xy$-plane or on the graph.

### 25.2 Tangent plane

The tangent plane to a surface, must contain 2 tangent lines. Think of pressing a board against the surface.

\[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \]

\textbf{Theorem:} (Assuming it’s not vertical) the equation of the tangent plane to the graph of $z = f(x, y)$ is

\[ \nabla f(x_0, y_0) \cdot (\Delta x, \Delta y) = 0. \]

\textbf{Proof:} If we hold $y$ constant, then one variable linear approximation gives

\[ \Delta z \approx f_x(x_0, y_0) \Delta x \quad \text{and} \quad \Delta y = 0. \]

That is, the displacement vector is approximately $\langle \Delta x, 0, f_x \Delta x \rangle$. So, the tangent vector to the curve (on the surface) $y = y_0, z = f(x, y_0)$ is $\langle 1, 0, f_x \rangle$.

Likewise holding $x$ fixed and varying $y$ gives the tangent vector $\langle 0, 1, f_y \rangle$.

Using the cross-product to get the normal we have the normal $\mathbf{N}$ is given by

\[ \mathbf{N} = \langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1). \]
So, the tangent plane has equation

$$-f_x \cdot (x - x_0) - f_y \cdot (y - y_0) + (z - z_0) = 0. \quad QED$$

### 25.2.1 Approximation formula

The tangent plane approximates surface, so we can use it to approximate $f(x, y)$

$$\Delta x = (x - x_0), \quad \Delta y = (y - y_0), \quad \Delta z = f(x, y) - f(x_0, y_0),$$

so

$$\Delta z \approx \frac{\partial z}{\partial x}_{x_0} \Delta x + \frac{\partial z}{\partial y}_{y_0} \Delta y.$$

Graph of $z = f(x, y)$.

**Example 25.3.** Suppose you have a box of dimensions 5, 10 and 15 cm. Use the tangent plane approximation formula to estimate the percentage change in volume if each of the dimensions is increased by 0.5 cm.

**answer:** Label the sides of the box, $x$, $y$, $z$.

Volume = $V = xyz$. So, $\frac{\partial V}{\partial x} = yz$; $\frac{\partial V}{\partial y} = xz$; $\frac{\partial V}{\partial z} = xy$.

So, $\frac{\partial V}{\partial x}|_{(5,10,15)} = 150$, $\frac{\partial V}{\partial y}|_{(5,10,15)} = 75$, $\frac{\partial V}{\partial z}|_{(5,10,15)} = 50$.

Therefore, the tangent plane approximation is

$$\Delta V \approx 150\Delta x + 75\Delta y + 50\Delta z.$$

We are given that $\Delta x = \Delta y = \Delta z = 0.5$. So,

$$\Delta V \approx (150 + 75 + 50) \cdot 0.5 = 137.5$$

This implies that the percentage change = $\Delta V/V = 137.5/750 = 18.3\%$.

**Example 25.4.** In the above example, volume is most sensitive to changes in which side?

**answer:** The side of length 5 since $\Delta x$ has the biggest coefficient in the approximation formula.