26 Tangent approximation, directional derivatives

26.1 Review of tangent plane approximation.

From the last topic we have the tangent plane approximation:

\[ \Delta x = (x - x_0), \quad \Delta y = (y - y_0), \quad \Delta w = f(x, y) - f(x_0, y_0) \]

\[ \Delta w \approx \frac{\partial w}{\partial x} \bigg|_{x_0} \Delta x + \frac{\partial w}{\partial y} \bigg|_{y_0} \Delta y \]

(The supplemental notes §TA give a nice analytic argument for this.)

In 3 dimensions we have \( w = f(x, y, z) \) and the tangent plane approximation is

\[ \Delta w = f(x, y) - f(x_0, y_0) \approx \frac{\partial w}{\partial x} \bigg|_{x_0} \Delta x + \frac{\partial w}{\partial y} \bigg|_{y_0} \Delta y + \frac{\partial w}{\partial z} \bigg|_{z_0} \Delta z. \]

26.2 Vector fields

Vector fields will be one of the key things we use in the rest of the course.

Definition. A vector field in two dimensions is a vector valued function of \((x, y)\), i.e.

\[ \mathbf{F}(x, y) = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} = \langle M(x, y), N(x, y) \rangle. \]

Often we will shorten this to \( \mathbf{F} = \langle M, N \rangle \).

Example 26.1. Here’s a vector field: \( \mathbf{F}(x, y) = \frac{x}{x^2 + y^2} \mathbf{i} + \frac{y}{x^2 + y^2} \mathbf{j} \). Identify \( M, N \) and sketch the vector field.
answer: \( M(x, y) = \frac{x}{x^2 + y^2}, N(x, y) = \frac{y}{x^2 + y^2} \).

We visualize it by drawing in the vector \( \mathbf{F}(x, y) \) at the point \((x, y)\) at various points in the plane. For example:

\[
\begin{align*}
\mathbf{F}(1, 0) &= \langle 1, 0 \rangle \\
\mathbf{F}(0, 1) &= \langle 0, 1 \rangle \\
\mathbf{F}(1, 1) &= \langle 1/2, 1/2 \rangle \\
\mathbf{F}(3, 0) &= \langle 1/3, 0 \rangle \\
\mathbf{F}(0, 2) &= \langle 0, 1/3 \rangle
\end{align*}
\]

We compute the field at a number of other points and draw the vectors in the plane. Note, \( \mathbf{F}(1, 1) \) is drawn from the point \((1, 1)\) etc.

The vector field \( \mathbf{F}(x, y) = \langle x/(x^2 + y^2), y/(x^2 + y^2) \rangle \).

Example 26.2. Here’s another vector field: \( \mathbf{F}(x, y) = 0.25y \mathbf{i} \). Identify \( M, N \) and sketch the vector field.

answer: \( M(x, y) = 0.25y, N(x, y) = 0. \)
The vector field \( F(x, y) = \langle 0.25y, 0 \rangle \).

Note that on the \( x \)-axis the vector field is 0.

Example 26.3. Sketch the constant vector field \( F = -0.5j = \langle 0, -1/2 \rangle \).

**Answer:** \( M(x, y) = 0, N(x, y) = -1/2 \).

The constant vector field \( F(x, y) = \langle 0, -1/2 \rangle \).

### 26.3 Gradient

For a function \( f(x, y) \) we know that \( \frac{\partial f}{\partial x} \) is the rate \( f \) changes when \( y \) is held constant and \( x \) changes. That is, the rate \( f \) changes as the point \((x, y)\) moves in the \( i \) direction.

Likewise \( \frac{\partial f}{\partial y} \) is the rate \( f \) changes as \((x, y)\) moves in the \( y \) direction.
We’ll see that it’s useful to put the partial derivatives together in a vector.

**Definition.** If \( w = f(x, y) \) the gradient of \( f \) at the point \((x_0, y_0)\) is

\[
\text{grad} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \mathbf{i} + \frac{\partial f}{\partial y}(x_0, y_0) \mathbf{j} = \left\langle \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle.
\]

Other notations for the gradient are

\[
\text{grad} f(x_0, y_0) = \nabla f(x_0, y_0) = \nabla w(x_0, y_0) = \nabla w|_o = \left\langle \frac{\partial w}{\partial x}|_o, \frac{\partial w}{\partial y}|_o \right\rangle.
\]

**Example 26.4.** Let \( f(x, y) = x^2 y + y^2 x + y^3 \). Compute the gradient of \( f \).

**Answer:** \( \nabla f = (2xy^2)\mathbf{i} + (x^2 + 2xy + 3y^2)\mathbf{j} = (2xy^2, x^2 + 2xy + 3y^2) \).

Note: \( f(x, y) \) is a scalar function and \( \nabla f \) is a vector field.

**Warning.** The function \( f(x, y) \) is a function of two variables. The graph if \( z = f(x, y) \) is a surface in 3 dimensional space, but the gradient \( \nabla f \) is a vector field in the plane. This moving between 3D and 2D can be confusing. The trick, as always, is to check dimensions on whatever you’re using.

### 26.3.1 The gradient field

Gradients of functions are one source of vector fields. We will see that an important question to ask about any vector field is whether or not it is the gradient of some function.

**Example 26.5.** Find the gradient vector field for the function \( f(x, y) = x^2 + y \).

**Answer:** \( \nabla f(x, y) = (2x, 1) \).

**Example 26.6.** Is the vector field \( \mathbf{F} = (y, x) \) a gradient field?

**Answer:** Later we’ll learn several ways to solve this problem. For now our only strategy is to notice that \( f(x, y) = xy \) has gradient \( \mathbf{F} \). So, the answer is yes, \( \mathbf{F} = \nabla xy \) is a gradient field.

### 26.4 Directional derivative

Remember that the \( \frac{\partial f}{\partial x} \) is the derivative of \( f \) as the point \((x, y)\) moves in the \( \mathbf{i} \) direction (i.e. \( y \) is not changing). Likewise for \( \frac{\partial f}{\partial y} \). What if \((x, y)\) moves in some other direction, so that both \( x \) and \( y \) are changing.

The directional derivative measure the rate change in other directions. Here are the ingredients:

1. A function \( z = f(x, y) \)
2. \( P_0 = (x_0, y_0) \) in the \( xy \)-plane.
3. A direction (unit vector) \( \mathbf{u} = (a, b) \) in the \( xy \)-plane.

**Directional derivative: geometric description.**

The geometric description of the directional derivative of \( f \) in the direction \( \mathbf{u} \) has a simple picture: Slice the graph of \( z = f(x, y) \) by a vertical plane through \( P_0 \) that contains \( \mathbf{u} \). The
intersection of the plane and the surface is a curve. The slope of the curve at the point above $P_0$ on the graph is the directional derivative, $\frac{df}{ds}\bigg|_{P_0, \hat{u}}$.

Two examples: directional derivative = slope of a vertical slice.

**NOTE WELL:** $P_0$ and $\hat{u}$ are in the plane.

**Directional derivative: analytic definition.**

The directional derivative of $f$ at $P_0$ in the direction $\hat{u}$ is defined as

$$\frac{df}{ds}\bigg|_{P_0, \hat{u}} = \lim_{\Delta s \to 0} \frac{\Delta z}{\Delta s}.$$  

Here $\Delta z$ is the change in $z$ caused by a step of length $\Delta s$ in the direction of $\hat{u}$ (all in the $xy$-plane). This is illustrated in the figure below.
The slope of the slice \( \frac{df}{ds} \bigg|_{P_0,\hat{u}} \approx \frac{\Delta z}{\Delta s} \).

**Example 26.7.** The applet [http://web.mit.edu/jorloff/www/jmoapplets/ddm-mathinsight/ddm-jmo.html](http://web.mit.edu/jorloff/www/jmoapplets/ddm-mathinsight/ddm-jmo.html) shows a topographic map of a stylized mountain. Move the point around on the level curves and see the corresponding point move on the mountain. The \( \theta \) sliders changes the direction of the green arrows. If you slide the red dot on the level curves in the direction of \( \hat{u} \) then the red dot moves up the mountain at slope \( \frac{df}{ds} \bigg|_{a,\hat{u}} \).

The applet uses the notation \( D_uf(a) \) for the directional derivative. Its value and a little green arrow showing whether it is positive or negative are shown on the applet.

### 26.4.1 Computing the directional derivative with the gradient

It turns out that we can compute directional derivatives using a simple formula with the gradient.

**Theorem.** If \( w = f(x,y) \) then the directional derivative is

\[
\frac{df}{ds} \bigg|_{P_0,\hat{u}} = \nabla f(P_0) \cdot \hat{u}.
\]

**Proof.** The key is the tangent plane approximation.

\[
\Delta z \approx \frac{\partial f}{\partial x} \bigg|_{P_0} \Delta x + \frac{\partial f}{\partial y} \bigg|_{P_0} \Delta y.
\]

The picture above illustrates two things:

1. \( \frac{df}{ds} \bigg|_{P_0,\hat{u}} \approx \frac{\Delta z}{\Delta s} \)
2. The vector \( \langle \Delta x, \Delta y \rangle \) is parallel to \( \hat{u} \).
By definition \( \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} = |\langle \Delta x, \Delta y \rangle| \). This means that (see picture below):

\[
\hat{u} = \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle \quad \text{(is a unit vector)}.
\]

Therefore, dividing the tangent plane approximation by \( \Delta s \) we get

\[
\frac{\Delta z}{\Delta s} \approx \frac{\partial f}{\partial x} \bigg|_{P_0} \frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y} \bigg|_{P_0} \frac{\Delta y}{\Delta s}
= \left\langle \frac{\partial f}{\partial x} \bigg|_{P_0}, \frac{\partial f}{\partial y} \bigg|_{P_0} \right\rangle \cdot \left\langle \frac{\Delta x}{\Delta s}, \frac{\Delta y}{\Delta s} \right\rangle
= \nabla f \bigg|_{P_0} \cdot \hat{u}. \quad \text{QED}
\]

In the limit the approximations becomes exactly the formula for the directional derivative in terms of the gradient.

**Example 26.8.** (Algebraic example) Let \( z = x^3 + 3y^2 \). Compute \( \frac{dw}{ds} \) at \( P_0 = (1, 2) \) in the direction of \( \mathbf{v} = 3\mathbf{i} + 4\mathbf{k} \).

**answer:** The gradient is \( \nabla z = \langle 3x^2, 6y \rangle \), so \( \nabla z \bigg|_{(1,2)} = \langle 3, 12 \rangle \).

The direction vector is \( \hat{u} = \langle 3/5, 4/5 \rangle \).

Therefore, the directional derivative is

\[
\frac{dz}{ds} \bigg|_{P_0, \hat{u}} = \nabla z \bigg|_{P_0} \cdot \hat{u} = \langle 3, 12 \rangle \cdot \langle 3/5, 4/5 \rangle = 57/5.
\]

**Example 26.9.** (Geometric example) Let \( \hat{u} \) be in the direction of \( \langle 1, -1 \rangle \). Using the picture below estimate \( \frac{\partial w}{\partial x} \bigg|_P, \frac{\partial w}{\partial y} \bigg|_P \), and \( \frac{dw}{ds} \bigg|_{P, \hat{u}} \).

**answer:** Measuring in the \( \mathbf{u} \) direction we get \( \Delta s \approx -.3 \). Therefore,

\[
\frac{dw}{ds} \bigg|_{P, \hat{u}} \approx \frac{\Delta w}{\Delta s} \approx \frac{10}{.3} = -33.3.
\]
Example 26.10. (Geometric example) Using the picture in the previous example, estimate $\frac{\partial w}{\partial x} \Bigg|_P$ and $\frac{\partial w}{\partial y} \Bigg|_P$.

**answer:** By measuring from $P$ to the next in level curve in the $x$ direction we see that $\Delta x \approx -0.5$. So, $\frac{\partial w}{\partial x} \Bigg|_P \approx \frac{\Delta w}{\Delta x} \approx \frac{10}{-0.5} = -20$.

Similarly, we get $\frac{\partial w}{\partial y} \Bigg|_P \approx 20$.

26.5 Direction of maximum change

The formula for the directional derivative gives us the following useful fact.

**Claim.** The gradient is the direction of the greatest rate of change. That is, at a point $P$, the direction $\hat{u} = \nabla f \Bigg|_P$ has the greatest directional derivative.

**Proof.** The formula for the directional derivative is

$$\frac{df}{ds} \bigg|_{P\hat{u}} = \nabla f \bigg|_P \cdot u.$$ 

We know the dot product has a maximum when the angle between the vectors is 0. That is, when they are in the same direction, i.e. $\hat{u} = \frac{\nabla f(P)}{|\nabla f(P)|}$.

Said in words: directional derivative = gradient-displacement.

This is maximized when displacement is parallel to the gradient. QED

26.6 The Gradient is perpendicular to level curves

The formula for the directional derivative gives us the following fact.

**Claim.** For $z = f(x, y)$, the gradient $\nabla f(P)$ is perpendicular to the level curve of $f$ through $P$. 
Proof. By definition the function does not change along a level curve. This means the directional derivative in the direction of the tangent to a level curve is 0.

So, if \( \hat{u} \) is tangent to the level curve we have

\[
\nabla f(P) \cdot \hat{u} = 0.
\]

That is, the gradient is perpendicular to \( \hat{u} \), i.e. perpendicular to the level curve.

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Example 26.11. Consider the graph of \( y = e^x \). Find a vector perpendicular to the tangent to at the point \((1, e)\).

Old method. Find the slope take the negative reciprocal and make the vector.

New method. This graph is the level curve of \( w = y - e^x \) with \( w = 0 \).

The gradient is \( \nabla w = (-e^x, 1) \). \( \Rightarrow \) normal = \( \nabla w(1, e) = (-e, 1) \).

Example 26.12. Sketch the level curves and gradient field for \( z = (x^2 + y^2)/3 \) and \( z = -(x^2 + y^2)/3 \).

26.7 Higher dimensions

For \( w = f(x, y, z) \) we get level surfaces. The gradient is normal to level surfaces.

Example 26.13. Find the tangent plane to the surface \( x^2 + 2y^2 + 3z^2 = 6 \) at the point \( P = (1, 1, 1) \).

answer: Introduce a new variable \( w = x^2 + 2y^2 + 3z^2 \). Our surface is the level surface \( w = 6 \), so the normal to the surface is \( \nabla w = (2x, 4y, 6z) \). At the point \( P \) we have \( \nabla w|_P = (2, 4, 6) \). Using point normal form the equation of the tangent plane is

\[
2(x - 1) + 4(y - 1) + 6(z - 1) = 0 \quad \text{or} \quad 2x + 4y + 6z = 12.
\]

Example 26.14. (Abstract version of previous example) Find the normal to the tangent plane of the graph of \( z = f(x, y) \) at \((x, y, z)\).
**Example 26.15.** (Not done in class) Let \( \theta = \tan^{-1}(y/x) \). Compute the directional derivative \( \frac{d\theta}{ds} \). Find the direction of maximum increase. Compute the directional derivative in this direction.

**answer:**
\[
\nabla \theta = -\frac{y}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j} = -\frac{y}{r^2} \hat{i} + \frac{x}{r^2} \hat{j}.
\]

The direction of maximum increase is \( \hat{u} = \text{dir} \nabla \theta \). So,
\[
\frac{d\theta}{ds} \bigg|_{\hat{u}} = |\nabla \theta| = \frac{1}{r}
\]

In this direction \( \frac{ds}{d\theta} = r \Rightarrow s = r\theta \) (around the circle)