29 Second derivative test; Lagrange multipliers

29.1 Second derivative test

For the function \( w = f(x, y) \), we know that a relative maximum or minimum occurs at a critical point. That is, a point where \( \nabla w = 0 \).

But not every critical point is a relative maximum or minimum. We have seen that they can also be saddles, or even none of the above. To distinguish which is which we use the second derivative test.

We’ll state the test first and see how it works, then we’ll show the reasoning behind it.

**Second derivative test.** Suppose \((x_0, y_0)\) is a critical point of \(w = f(x, y)\). Compute (and name) all three second derivatives.

\[ A = w_{xx}(x_0, y_0), \quad B = w_{xy}(x_0, y_0), \quad C = w_{yy}(x_0, y_0) \]

Then:
- If \( AC - B^2 > 0 \), and if \( A > 0 \) then the critical point \((x_0, y_0)\) is a minimum;
- If \( AC - B^2 > 0 \), and if \( A < 0 \) then the critical point \((x_0, y_0)\) is a maximum;
- If \( AC - B^2 < 0 \) then \((x_0, y_0)\) is a saddle;
- If \( AC - B^2 = 0 \) then the test fails.

**Example 29.1.** For \( w = x^3 - 3xy + y^3 \) find and classify all the critical points.

**Answer:** Computing first derivatives: \( w_x = 3x^2 - 3y, \ w_y = -3x + 3y^2 \)

Critical points: (solve the simultaneous equations)

First: \( 3x^2 - 3y = 0 \) \(\Rightarrow\) \( y = x^2 \).

Substitute this into \( -3x + 3y^2 = 0 \) \(\Rightarrow\) \( x^4 - x = 0 \) \(\Rightarrow\) \( x(x^3 - 1) = 0 \) \(\Rightarrow\) \( x = 0, 1 \).

So, the critical points are \((0, 0), (1, 1)\).

Compute second derivatives: \( w_{xx} = 6x, \ w_{xy} = -3, \ w_{yy} = 6y \).

At \((0, 0)\): \( A = w_{xx} = 0, \ B = w_{xy} = -3, \ C = w_{yy} = 0 \), so \( AC - B^2 = -9 < 0 \).

The second derivative test implies \((0, 0)\) is a saddle.

At \((1, 1)\): \( A = 6, \ B = -3, \ C = 6 \), so \( AC - B^2 = 27 \). Since \( A > 0 \), the second derivative test implies \((1, 1)\) is a minimum.

29.1.1 Reasoning behind the second derivative test

The reasoning uses the second order approximation formula. For simplicity we’ll assume the critical point is at \((0, 0)\). Translating to a different point is not hard.

Let \( w_0 = w(0, 0) \). The second order approximation is

\[
w \approx w_0 + \left. \frac{\partial w}{\partial x} \right|_0 x + \left. \frac{\partial w}{\partial y} \right|_0 y + \frac{1}{2} \left. \frac{\partial^2 w}{\partial x^2} \right|_0 x^2 + \left. \frac{\partial^2 w}{\partial x \partial y} \right|_0 xy + \frac{1}{2} \left. \frac{\partial^2 w}{\partial y^2} \right|_0 y^2.
\]
Challenge. Recall why this is the second order approximation.

Writing $\Delta w = w - w_0$, the approximation is

$$\Delta w \approx \left. \frac{\partial w}{\partial x} \right|_0 x + \left. \frac{\partial w}{\partial y} \right|_0 y + \frac{1}{2} \left. \frac{\partial^2 w}{\partial x^2} \right|_0 x^2 + \frac{1}{2} \left. \frac{\partial^2 w}{\partial y^2} \right|_0 y^2 + \frac{1}{4} \left. \frac{\partial^2 w}{\partial x \partial y} \right|_0 xy + \frac{1}{4} \left. \frac{\partial^2 w}{\partial x^2} \right|_0 x^2 + \frac{1}{4} \left. \frac{\partial^2 w}{\partial y^2} \right|_0 y^2.$$

At a critical point $w_x = 0$ and $w_y = 0$, so this becomes

$$\Delta w = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2),$$

where $A$, $B$ and $C$ are as defined in the second derivative test.

Now complete the square:

$$\Delta w = \frac{A}{2} \left( \left( x + \frac{B}{A}y \right)^2 + \frac{(AC - B^2)y^2}{A^2} \right).$$

Because squares are always positive we can determine the sign of $\Delta w$ in the various cases.

If $AC - B^2 > 0$ and $A > 0$ then both terms in $\Delta w$ are positive, so $\Delta w > 0$. This implies $(x_0, y_0)$ is a minimum.

If $AC - B^2 > 0$ and $A < 0$ then $\Delta w < 0$. This implies it’s a maximum.

If $AC - B^2 < 0$ then $\Delta w$ can be both positive or negative, so we have a saddle.

### 29.2 Canonical second order examples

**Example 29.2.** (Use these to help in remembering the rules.)

(i) Let $z = x^2 + y^2$. This has a minimum at $(0,0)$. Computing second derivatives we have $A = 2, B = 0, C = 2$. So, $AC - B^2 = 4 > 0$ and $A > 0$. Thus, the second derivative test agrees there is a minimum at $(0,0)$.

(ii) Let $z = -x^2 - y^2$. This has a maximum at $(0,0)$. Computing second derivatives we have $A = -2, B = 0, C = -2$. So, $AC - B^2 = 4 > 0$ and $A < 0$. Thus, the second derivative test agrees there is a maximum at $(0,0)$.

(iii) Let $z = y^2 - x^2$. This has a saddle at $(0,0)$. We have $A = -2, B = 0, C = 2$, so $AC - B^2 = -4 < 0$. The test agrees there is a saddle at $(0,0)$.

(iv) Let $z = xy$. This has a critical points at $(0,0)$. Computing second derivatives we have $A = 0, B = 1, C = 0$, so $AC - B^2 = -1 < 0$. The second derivative test tells us this is a saddle.

**Example 29.3. General second order example**

Let $z = \frac{1}{2}(ax^2 + 2bxy + cy^2)$. This has a critical point at $(0,0)$.

It’s easy to compute: $A = a, B = b, C = c$. Not surprisingly, $AC - B^2 = ac - b^2$.

**Example 29.4.** Use the second derivative test to try to determined the type of critical point at $(0,0)$ of $z = x^5 + y^4 - 5x - 32y - 7$.

**answer:** This is a straightforward computation:
\[ z = 5x^4 - 5, \quad z_y = 4y^3 - 32, \quad z_{xx} = 20x^3, \quad z_{yy} = 12y^2, \quad z_{yx} = 0. \]

Therefore \( A = z_{xx}(0,0) = 0, \quad B = z_{yx}(0,0) = 0, \quad C = z_{yy}(0,0). \) So, \( AC - B^2 = 0 \) and the test gives no information.

### 29.3 Lagrange multipliers

Lagrange multipliers is a technique for finding maxima and minima of a function with constraints.

**Example 29.5.** Find the maximum value of \( z = x^2 - y^2 \) subject to the constraint \( x^2 + y^2 = 1. \)

**Answer:** Notice that without the constraint \( z = x^2 - y^2 \) has no maximum. Geometrically the constraint says we will only consider points \((x,y)\) on the unit circle. It’s easy to see that the maxima of \( z \) on the circle are at \((1,0)\) and \((0,1)\). Likewise, the minima are at \((0,1)\) and \((0,-1)\).

![Graph showing maxima and minima](image)

Max. and min. for \( x^2 - y^2 \) constrained to \( x^2 + y^2 = 1. \)

### 29.3.1 General Lagrange multipliers question

The general constrained min-max problem is:

Find the minimum (maximum) of the function \( w = f(x,y) \) constrained by \( g(x,y) = c \) (\( c \) is a constant).

The function \( f \) is called the objective function and \( g(x,y) = c \) is called the constraint equation.

We need the definition of constrained critical points.

**Definition.** In this setting, a constrained critical point is one where \( \nabla f(x,y) = \lambda \nabla g(x,y) \) and \( g(x,y) = c. \) Here, \( \lambda \) is some (unknown) multiplier.

The Lagrange multipliers solution is: Constrained relative minima and maxima occur at constrained critical points.

There is no problem extending this to functions of more variables, like \( w = f(x,y,z) \) with the constraint \( g(x,y,z) = c. \)

**Example 29.6.** We want to build a box out of cardboard. The box will have no top, a single thickness of cardboard on the front and back, double thick sides, and a triple thick bottom. We require the volume \( = 3. \)

What’s the smallest amount of cardboard you can use?
answer: We did this problem in the previous topic by using the constraint to eliminate one variable. Here we’ll use Lagrange multipliers.

The dimensions of the box are $x$, $y$, $z$.
The front and back each have area $yz$. The sides have area $xz$ and the bottom has area $xy$. Taking the thicknesses into account we have the following constrained min-max problem. Minimize $w = 2xz + 4yz + 3xy$, subject to the constraint $V = xyz = 3$.

**answer:** Compute the gradients: $\nabla w = (2z + 3y, 4z + 3x, 2x + 4y)$, $\nabla V = (yz, xz, xy)$

Lagrange multiplier equations: $(2z + 3y, 4z + 3x, 2x + 4y) = \lambda(yz, xz, xy)$, and $xyz = 3$. Let’s write these out individually:

\[
\begin{align*}
2z + 3y &= \lambda yz \\
4z + 3x &= \lambda xz \\
2x + 4y &= \lambda xy \\
xyz &= 3
\end{align*}
\]

We solve symmetrically, i.e. isolate $\lambda$ and eliminate it.

\[
\begin{align*}
2/y + 3/z &= \lambda \\
4/x + 3/z &= \lambda \\
2/y + 4/x &= \lambda \\
xyz &= 3
\end{align*}
\]

Thus, $y = x/2$ and $z = 3x/4$. Putting this into the constraint equation we get $3x^3/8 = 3$. So, $x = 2$.

The dimensions that use the least cardboard are $x = 2, y = 1, z = 3/2$.

**Example 29.7.** (Sphere example) Minimize $w = y$ constrained to $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

**answer:** It should be clear that the minimum occurs when $y = -1, x = 0, z = 0$. But, let’s see this using Lagrange multipliers.
\[ \nabla w = \langle 0, 1, 0 \rangle, \quad \nabla g = \langle 2x, 2y, 2z \rangle. \] So, the Lagrange multiplier equations are

\[
\begin{align*}
0 &= \lambda 2x \\
1 &= \lambda 2y \\
0 &= \lambda 2z \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]

It’s clear the solutions to this require \( x = 0 \) and \( z = 0 \). So, the constraint implies we must have \( y = \pm 1 \). Evaluating \( w \) at \( (0, 0, \pm 1) \) shows that \( (0, 0, -1) \) is a minimum with \( w = -1 \) and \( (0, 0, 1) \) is a maximum with \( w = 1 \).

### 29.4 Reason for Lagrange

#### 29.4.1 Geometric argument

We’ll do this using a two dimensional picture. The idea is the same in 3 dimensions, but harder to draw.

Suppose \( w = f(x, y) \) and we have the constraint \( g(x, y) = c \). We want to show that constrained relative maxima (and minima) occur at constrained critical points. That is, at points where \( \nabla f \) is a multiple of \( \nabla g \).

In the figure below the orange circle is the constraint curve and the blue arcs are the level curves \( w = f(x, y) = \text{constant} \).

We will find the maximum dynamically. Since we are constrained to lie on the orange circle, suppose we start at the point \( A \). At \( A \) we are on the level curve \( w = 1 \). Since we want a maximum we move to a level curve with bigger \( w \). We have to stay on the circle, so we end up at \( B \), with \( w = 2 \). We can keep doing this until we get to \( C \), with \( w = 3 \). At this point, the next bigger level doesn’t touch the constraint circle, so we can’t get there, e.g., the \( w = 4 \) curve does not intersect the constraint.

The last level curve to touch the constraint circle has to be tangent to it. That is, their normals must be parallel. That is, the two gradients must be parallel. \( \quad \text{QED} \)

**Challenge.** Redo this argument in 3 dimensions for the sphere example above. Hint, the level surfaces of \( w = y \) are vertical planes parallel to the \( xz \) plane.
29.4.2 Analytic argument

This doesn’t require pictures, so we’ll do it in 3 dimensions.

The constraint $g(x, y, z) = c$ is a level surface with normal $\nabla g$.

Suppose $P_0$ is a relative minimum for $f$ on the surface.

Let $r(t)$ be any curve on the surface with $r(0) = P_0$.

This implies, $h(t) = f(r(t))$ has a relative minimum at $t = 0$.

Taking a derivative: $h'(t) = \nabla f |_{r(t)} \cdot r'(t)$.

Since $t = 0$ is a minimum, we must have $h'(0) = \nabla f |_{P_0} \cdot r'(0) = 0$.

So, $\nabla f |_{P_0}$ is perpendicular to any curve on the surface through $P_0$. That is, $\nabla f |_{P_0}$ is normal to the surface, i.e. it is parallel to $\nabla g |_{P_0}$. QED

29.5 Need to check the boundary

If you are looking for the absolute maximum of a function it can occur at a relative maximum or at one of the edges or corners of the region where you are looking for the maximum. Of course, the same applies to minima. We’ll illustrate with some examples.

Example 29.8. As a first example, let’s remind ourselves how this works in 18.01. Consider the function $f(x)$ shown over the interval $[a, b]$. There is only one critical point and we can see that it’s the absolute maximum. The absolute minimum is not at a critical point. Instead, it occurs at $x = b$, which is at the end of the interval we are considering.

Check the boundary for absolute maxima or minima.

Example 29.9. (Checking the boundary)

A rectangle in the plane is placed in the first quadrant so that one corner is at the origin $O$ and the two sides adjacent to $O$ are on the axes. The corner $P$ opposite $O$ is on the curve $x + 2y = 1$. Using Lagrange multipliers find for which point $P$ the rectangle has maximum area. Say how you know this point gives the maximum.

**Answer:** We need some names. Let $g(x, y) = x + 2y = 1 = \text{constraint equation}$ and let area $= f(x, y) = xy$ be the objective function.

Computing gradients: $\nabla g = (1, 2), \, \nabla f = (y, x)$.

The Lagrange multiplier equations are

\begin{align*}
y &= \lambda \\
x &= 2\lambda \\
x + 2y &= 1
\end{align*}
The first two equations imply $x = 2y$. Combine this with the third equation to get $4y = 1$. So, $y = 1/4, \ x = 1/2 \Rightarrow P = (1/2, 1/4)$.

The vertex $P$ opposite $O$ is constrained to the orange line.

We know this is a maximum because the maximum occurs either at a critical point or on the boundary. In this case the boundary points are on the axes which gives a rectangle with area $= 0$.

**Example 29.10.** (Boundary at $\infty$)

A rectangle in the plane is placed in the first quadrant so that one corner $O$ is at the origin and the two sides adjacent to $O$ are on the axes. The corner $P$ opposite $O$ is on the curve $xy = 1$. Using Lagrange multipliers find for which point $P$ the rectangle has minimum perimeter. Say how you know this point gives the minimum.

**answer:** Let $g(x, y) = xy = 1 = \text{constraint}$ and $f(x, y) = 2x + 2y = \text{perimeter}$.

Gradients: $\nabla g = \langle y, x \rangle, \ \nabla f = \langle 2, 2 \rangle$.

The Lagrange multiplier equations are

\[
2 = \lambda y \\
2 = \lambda x \\
xy = 1
\]

The first two equations imply $x = y$. Combine this with the third equation to get $x^2 = 1$. Thus, $x = 1, \ y = 1$. So there is one constrained critical point at $P = (1, 1)$.

The point $P$ is constrained to the orange hyperbola.

We know this is a minimum because the minimum occurs either at a critical point or on the boundary. In this case the boundary points are infinitely far out on the axes which gives a rectangle with perimeter $= \infty$. 