4 Definite integrals; fundamental theorem

**Definition:** The definite integral is the area between the graph and the $x$-axis. This is denoted with an elongated 'S' as shown in the caption to the figure below.

![Definite Integral Graph](image)

$$\int_a^b f(x) \, dx = \text{area under curve} = \text{the integral of } f(x) \text{ from } a \text{ to } b$$

**Examples:** For simple graphs we can compute the area directly:

4.1. Compute the definite integral $\int_0^1 x \, dx$.

**answer:** $\int_0^1 x \, dx = \text{shaded area in the figure below} = \frac{1}{2}$.

![Simple Graph](image)

Definite integral = area under $y = x$, over $x$-axis, between $x = 0$ and $x = 3$.

**Dummy variables:** We can use any symbol for the variable used in integration. That is

$$\int_0^1 x \, dx = \int_0^1 u \, du = \int_0^1 t \, dt \quad \text{or} \quad \int_a^b f(x) \, dx = \int_a^b f(u) \, du$$

We often refer to the variable used in integration as a **dummy variable** because it’s just there for integration and doesn’t play a role in the rest of a problem or argument.

**Area below the $x$-axis counts negative:** If the curve is below the $x$-axis, i.e. the value of the function is negative, then we will count the area as negative in the integral.

Example 4.2.
4.1 Summation notation

We will often use sums with many terms that show a pattern.

Example 4.3.

1. $1 + 2 + 3 + \ldots + 999 + 1000$.
2. $1 + 1/2 + 1/3 + \ldots + 1/999 + 1/1000$.
3. $1^2 + 2^2 + \ldots + N^2$.

The ellipsis indicates we didn’t write down every term. Often this is okay since we can see the pattern. But, this notation is not always clear. One way to be fully specific and to be more compact is to use summation notation.

Here are the previous examples written in summation notation.

1. $1 + 2 + \ldots + 1000 = \sum_{n=1}^{1000} n$.
2. $1/1 + 1/2 + \ldots + 1/1000 = \sum_{n=1}^{1000} 1/n$.
3. $1 + 2^2 + \ldots + N^2 = \sum_{n=1}^{N} n^2$.

The letter $\sum$ is the uppercase Greek letter sigma –for summation.

In the examples, the letter $n$ is the index and the terms above and below the $\sum$ are the limits of the sum. The formula $\sum_{n=1}^{1000} n^2$ is read as ’the sum from $n = 1$ to 1000 of $n^2$.'
This is no real mystery to this notation. If it’s hard to understand a formula just write it out long-hand.

**Example 4.4.** Compute \( \sum_{j=2}^{5} j^2 \).

**Answer:** \( \sum_{j=2}^{5} j^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54 \). Notice that to get this we just marched the index \( j \) along from 2 to 5.

Once you see the pattern you can write a series using summation notation.

**Example 4.5.** Write \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + 100 \cdot 101 \) in summation notation.

**Answer:** \( \sum_{k=1}^{100} k \cdot (k + 1) \).

**Example 4.6.** Write the sum from \( k = 7 \) to 23 of \( \sin(k\pi/100) \) in summation notation.

**Answer:** \( \sum_{k=7}^{23} \sin(k\pi/100) \).

### 4.2 Computing areas under curves by the method of Exhaustion

(We exhaust the area, hopefully you’ll be exhilarated.) This is one of the main points in 18.01A. We will use it many, many times to set up integrals. The idea is to approximate the region by a lot of thin rectangles and approximate the area by summing the area of all the rectangles.

In this section we will look at an explicit example using \( y = x^2 \). Later we will work abstractly with a general function \( y = f(x) \).

**Example 4.7.** Approximate the integral \( \int_{1}^{2} x^2 \, dx \) by approximating the area with 2 rectangles. Repeat this with 4 and then 8 rectangles.

**Answer:** The figures below covers the region under the curve by 2, 4 and 8 rectangles respectively.

The area under \( y = x^2 \) covered by 2, 4 and 8 rectangles.

We go through the cases one at a time.

**2 rectangles:** The width of each rectangle is 0.5. The height of each rectangle is the height of the curve at the right-side of the rectangle, e.g. the right side of first rectangle is...
at \( x = 1.5 \) and the rectangle has height \( y = (1.5)^2 \).

We approximate the area under the curve by summing the area of the rectangles:

\[
\int_{1}^{2} x^2 \, dx \approx (0.5)(1.5)^2 + (0.5)(2)^2 = 3.125.
\]

Since both rectangles stick up above the curve this is clearly an over estimate.

**4 rectangles:** The middle figure covers the region under the curve by 4 rectangles. The width of each rectangle is now 1/4 and the heights are again given by the height of the curve at the right-side of the rectangle.

Summing the area of the rectangles:

\[
\int_{1}^{2} x^2 \, dx \approx (0.25)(1.25)^2 + (0.25)(1.5)^2 + (0.25)(1.75)^2 + (0.25)(2)^2 = 2.7185.
\]

**8 rectangles:** The third figure covers the region under the curve by 8 rectangles. Each rectangle has width 1/8. (We use fractions instead of decimals simply to fit them in the picture.)

Summing the area of the rectangles:

\[
\int_{1}^{2} x^2 \, dx \approx \frac{1}{8} \cdot \left( \frac{9}{8} \right)^2 + \frac{1}{8} \cdot \left( \frac{10}{8} \right)^2 + \frac{1}{8} \cdot \left( \frac{11}{8} \right)^2 + \frac{1}{8} \cdot \left( \frac{12}{8} \right)^2 + \frac{1}{8} \cdot \left( \frac{13}{8} \right)^2 + \frac{1}{8} \cdot \left( \frac{14}{8} \right)^2 + \frac{1}{8} \cdot \left( \frac{15}{8} \right)^2 + \frac{1}{8} \cdot 2^2 \\
\approx 2.523
\]

**More, thinner rectangles gives a better estimate.** Notice that in all of the figures the rectangles stick up over the curve and so our estimates are all overestimates. Also notice that as we use more, thinner rectangles they do a better and better job of approximating the region under the curve.

We call this approach the **method of exhaustion** because as we increase the number of rectangles the error in the estimate goes to 0, that is we ‘exhaust’ the error.

### 4.3 The method of exhaustion for arbitrary functions

Here are the steps for estimating the definite integral, \( \int_{a}^{b} f(x) \, dx \) of a function \( f(x) \). Don’t forget that the definite integral is just the area under the graph of \( f(x) \).

1. Divide \([a, b]\) into \( n \) equal intervals. So each interval has width \( \Delta x = \frac{b - a}{n} \)
2. Pick any point \( c_i \) in the \( i \)-th interval.
3. The height of the \( i \)-th rectangle is \( f(c_i) \), i.e. the height of the curve at \( x = c_i \).
4. Given the width and height we know the area of the \( i \)-th rectangle is

\[
\text{Area of } i\text{-th rectangle} = f(c_i) \Delta x
\]
5. Approximate the area under the curve by summing the area of the rectangles:

\[
\int_{a}^{b} f(x) \, dx = \text{area under curve} \\
\approx \text{sum of area of rectangles} \\
= \sum_{1}^{n} f(c_i) \Delta x
\]

The following figure shows all the pieces of this algorithm.

We will call the sum \( \sum_{1}^{n} f(c_i) \Delta x \) a Riemann Sum.

To summarize:
- The Riemann sum approximates the integral

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{1}^{n} f(c_i) \Delta x
\]
- In the limit as the number of rectangles \( n \) goes to \( \infty \) the Riemann sum goes to the value of the integral, i.e. the true area under the curve.
- As \( n \to \infty \) the width of each rectangle \( \Delta x \to 0 \).

4.4 How to choose \( c_i \)

The general algorithm is intentionally vague about how to choose the point \( c_i \) in each interval. This is because we can do it any way we want. We won’t give a formal proof, but this should seem plausible, since as the rectangles get thinner the height is basically the same for any choice of \( c_i \). For the record we state this as a theorem

**Theorem.** No matter how you choose \( c_i \) in the limit as \( n \to \infty \) the Riemann sum goes to the value of the integral.

**Proof.** One proof uses the mean value theorem. You can find it in the textbook.

There are many ways to **choose the point \( c_i \) in each interval.** Typical choices are:
- Choose the left endpoint of each interval
4 Definite integrals; fundamental theorem

- Choose the right endpoint if each interval
- Choose the midpoint of each interval
- Choose the point in each interval that maximizes the height of the rectangle.
- Choose the point in each interval that minimizes the height of the rectangle.
- Choose a random point in each interval.

The corresponding Riemann sums are the called respectively the left, right, mid, upper, lower and random Riemann sum.

**Example 4.8.** Approximate \( \int_0^1 x \, dx \) using a right Riemann sum. (Give the general formula for \( n \) rectangles.)

**Answer:** The figure shows \([0,1]\) divided into \( n \) equal intervals.

The right endpoints are \( x_1 = 1/n, x_2 = 2/n, \ldots x_i = i/n, \ldots \). The height of the \( i \)-th rectangle is \( f(x_i) = i/n \). Therefore

\[
\int_0^1 x \, dx \approx \sum_{i=1}^{n} \left( \frac{i}{n} \right) \cdot \frac{1}{n} \quad (\text{Here } \Delta x = \frac{1}{n}, \text{ right endpt } = \frac{i}{n}).
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1+1/n}{2}.
\]

In the limit as \( n \to \infty \) we see the Riemann sum becomes \( \frac{1}{2} \), which is the area under the curve.

**Note:** The textbook by Simmons computes \( \int x \, dx, \int x^2 \, dx, \int x^4 \, dx \). The computation relies on formulas for \( \sum i, \sum i^2, \sum i^4 \).

### 4.5 First Fundamental Theorem of Calculus

The definite integral is defined as an area. So far our only method of computing it is to use the rather tiring 'method of exhaustion'. Fortunately there is a much easier way to compute integrals.

**Theorem. (The first fundamental theorem of calculus)** If \( f(x) \) is continuous and \( F'(x) = f(x) \) then

\[
\int_a^b f(x) \, dx = F(b) - F(a) = \text{ (definition) } F(x)|_a^b.
\]
This says that finding the area is equivalent to finding an anti-derivative. This is a BIG idea.

Notes: 1. In the box we introduced a shorthand notation, instead of \( F(b) - F(a) \) we can write \( F(x)|_a^b \). They mean the same thing.

2. When you see something called ‘The Fundamental Theorem’ you should assume it’s important. In this case, it warrants a lot of attention and we will sketch three proofs.

Example 4.9. (We can’t possibly do all these in class.)

1. Use the first fundamental theorem to compute \( \int_0^1 x^3 \, dx \)

\[
\text{answer: } \int_0^1 x^3 \, dx = \frac{x^4}{4} \bigg|_0^1 = \frac{1}{4} \text{ (draw your own picture).}
\]

2. Same question for \( \int_0^{\pi/a} \sin ax \, dx \).

\[
\text{answer: } \int_0^{\pi/a} \sin ax \, dx = -\frac{1}{a} \cos ax \bigg|_0^{\pi/a} = \frac{2}{a} \text{. (Note the picture below shows us that the integral is positive –it’s easy to mess up signs.)}
\]

3. Show \( \int_0^{2\pi} \sin x \, dx = 0 \).

\[
\text{answer: } \int_0^{2\pi} \sin x \, dx = -\cos x \bigg|_0^{2\pi} = 0.
\]

4. Compute \( \int_1^2 \frac{1}{x} \, dx \).

\[
\text{answer: } \int_1^2 \frac{1}{x} \, dx = \ln |x| \bigg|_1^2 = \ln 2.
\]

5. Given a rod of length 2 m with density \( \delta(x) = 2 - (x - 1)^2 \) g/m, find the total mass of the rod. (Be sure you understand why it’s okay to speak of density of a rod in mass/length.)

\[
\text{answer: } \text{This is a problem where we first need to ‘build’ the integral as a Riemann sum: To do this we divide the rod into } n \text{ small segments:}
\]

The figure shows a rod of length two divided into \( n \) equal segments, a point \( c_i \) is chosen in each segment. As usual we’ll call the length of each segment \( \Delta x \).

Now, the key step in building integrals is the following observation: If the \( i \)-th segment
is small enough then the density of the segment is essentially constant and equal to \( \delta(c_i) \).
This means that the mass of the \( i \)-th segment is approximately \( \delta(c_i) \Delta x \). Adding up the mass of each segment we get

\[
\text{mass of the rod} = M \approx \sum_{i=1}^{n} \delta(c_i) \Delta x.
\]

This is a Riemann sum! In the limit as \( n \to \infty \) the sum becomes an integral and we get the exact formula:

\[
M = \int_{0}^{2} \delta(x) \, dx = \int_{0}^{2} 2 - (x - 1)^2 \, dx = 2x - (x - 1)^3 / 3 \bigg|_{0}^{2} = 4 - 2/3 = 10/3.
\]

4.6 An important convention

So far we’ve always had \( a < b \), the following will be quite useful.

\[
\int_{b}^{a} f(x) \, dx = - \int_{a}^{b} f(x) \, dx.
\]

4.7 Properties of definite integrals

1. \( \int_{a}^{b} f(x) + g(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \).
2. \( \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx \).
3. \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \).
4. \( \left| \int_{a}^{b} f(x) \, dx \right| \leq \int_{a}^{b} |f(x)| \, dx \).

(All of these properties follow from the definition of integral as area.)

Example 4.10. (properties 1 and 3)
\[
\int_{1}^{2} 3x^3 + 4x \, dx = 3 \int_{1}^{2} x^3 \, dx + 4 \int_{1}^{2} x \, dx = 3x^4/4\bigg|_{1}^{2} + 4x^2/2\bigg|_{1}^{2} = 17 - 1/4 = 16.75.
\]

Example 4.11. (property 2)
\[
\int_{-1}^{1} x^3 \, dx = \int_{-1}^{0} x^3 \, dx + \int_{0}^{1} x^3 \, dx = x^4/4\bigg|_{-1}^{0} + x^4/4\bigg|_{0}^{1} = -1/4 + 1/4 = 0.
\]

Example 4.12. (property 2)
\[
\int_{-1}^{1} |x| \, dx = \int_{-1}^{0} |x| \, dx + \int_{0}^{1} |x| \, dx = \int_{-1}^{0} -x \, dx + \int_{0}^{1} x \, dx = -x^2/2\bigg|_{-1}^{0} + x^2/2\bigg|_{0}^{1} = 1/2 + 1/2 = 1.
\]

Example 4.13. (property 4)
\[
\int_{-1}^{1} x^3 \, dx = x^4/4\bigg|_{-1}^{1} = 0.
\]
\[ \int_{-1}^{1} |x^3| \, dx = \int_{-1}^{0} -x^3 \, dx + \int_{0}^{1} x^3 \, dx = 1/2 \Rightarrow \left| \int_{-1}^{1} x^3 \, dx \right| \leq \int_{-1}^{1} |x^3| \, dx. \]

For each of these examples you should be able to draw a picture and understand the algebraic manipulations in terms of areas.

### 4.8 Proofs of the first fundamental theorem

Here are the three promised proofs of the first fundamental theorem.

**Proof 1**: (velocity and distance) We’ll prove the fundamental theorem by thinking of the function \( F(t) \) as the position of a mass moving along a line. Then \( F'(t) = f(t) \) is the velocity of the mass and \( F(b) - F(a) \) is the net displacement over the time interval \([a, b]\).

(We use displacement because if the mass starts and ends at the same place its displacement \( F(b) - F(a) = 0 \).)

The fundamental theorem says that \( \int_{a}^{b} f(t) \, dt = F(b) - F(a) \). In words it says that the integral of velocity is displacement. We’ll show that this is the case by building the integral from a Riemann sum.

Divide \([a, b]\) into \( n \) equal intervals and choose a value \( c_i \) in each interval. Over a small time slice the velocity is approximately constant and we can compute the distance traveled as velocity \( \times \) time. If this is negative the displacement is to the left and if it’s positive the displacement is to the right.

Thus in the \( i \)-th interval the distance traveled is approximately \( f(c_i) \Delta t \).

\[
\begin{array}{cccccccc}
& & t_0 & c_1 & t_1 & c_2 & t_2 & c_3 & \cdots & t_{n-1} & c_n & t_n & b \\
& & a & & & & & & & \\
\end{array}
\]

The net displacement is the sum of the displacements over all the small intervals. Thus

\[
\text{Net displacement} = F(b) - F(a) \approx \sum_{i=1}^{n} f(c_i) \Delta t.
\]

The sum on the right is a Riemann sum. If we let the number of intervals \( n \) go to infinity then the Riemann sum becomes an integral and the approximation becomes exact. That is

\[
\text{Net displacement} = F(b) - F(a) = \int_{a}^{b} f(t) \, dt
\]

This is the formula we wanted to prove, so we are done!

**Proof 2**: (Mean Value Theorem)

\[
F(b) - F(a) = \sum_{i=1}^{n} F(t_i) - F(t_{i-1})
= \sum_{i=1}^{n} F'(c_i)(t_i - t_{i-1}), \quad \text{where } c_i \text{ is from the MVT}
= \sum_{i=1}^{n} f(c_i) \Delta t.
\]

As always, this sum \( \to \int_{a}^{b} f(t) \, dt \) as \( \Delta t \to 0 \).
We won’t mention this proof again.

**proof 3:** See the textbook by Simmons §6.6

### 4.9 Sums approximate integrals and integrals approximate sums

So far we have use Riemann sums to approximate integrals. We can turn this around and use integrals to approximate sums. The reason to do this is that sums can be hard to compute, but, thanks to the fundamental theorem, integrals are often easy to compute.

**Example 4.14.** A suspension bridge of length 100 m. has a parabolic support given by

\[ f(x) = x(100 - x) \]

The vertical steel cables are placed 1/2 meter apart. Approximate the total length of cable used.

**answer:** The exact length of cable is

\[ L = f(1/2) + f(1) + f(3/2) + \ldots + f(100) = \sum_{n=1}^{200} f(n/2). \]

This is a sum, which we will approximate by an integral.

It is typical that what we want to compute is not exactly something we know but can be manipulated into a form we recognize. As long as we keep track of the manipulations we can then go backwards and compute what we want.

In this case the sum for \( L \) is close to a Riemann sum for \( \int_0^{100} f(x) \, dx \). So, if we divide the interval \([0, 100]\) into 200 subintervals then \( \Delta x = 1/2 \) and we have the right Riemann sum.

\[ \int_0^{100} f(x) \, dx \approx \frac{1}{2} \sum_{n=1}^{200} f(n/2). \]

The sum on the right hand side is just 1/2 the sum for \( L \), so we have

\[ L \approx 2 \int_0^{100} f(x) \, dx = 2 \left[ \frac{50x^2 - x^3}{3} \right]_0^{100} = \frac{10}{3} \times 10^5 \approx 3.33 \times 10^5. \]

(Using a computer we find the exact value is 333,325.)