18.02a Problem Set 7
(due in class on Thursday, Nov. 9)

Part I  (22 points)

TB = Simmons; SN = 18.02A Supplementary Notes (all have solutions)
The problems marked 'other' are not to be handed in.

Topic 22 (R, Nov. 2)  Parametric equations continued.
Read: TB: 17.4
Hand in: 1J/1ac, 3, 4ab, 5, 6, 9abc, find the curvature of the helix in 1J/6.

Topic 23 (Not done in class)  Kepler’s second law.
Read: SN: K
Hand in: None

Topic 24 (M, Nov. 6)  Functions of several variables, partial derivatives.
Read: TB: 19.1
Hand in: 2A/1abe

Topic 25 (T, Nov. 7)  Tangent plane, level curves, contour surfaces.
Read: TB: 19.2  SN: TA

Continuation: (W, Nov. 8)  Discussion, review and catch up.
Continuation: (R, Nov. 9)  Discussion, review and catch up.

Coming next

Exam: (M, Nov. 13)  Exam 3 (covers 17-24)

Part II  (96 points)

Problem 1 (Class 22: 20 pts: 10,5,5)
(a) Find the unit tangent vector, unit normal vector, radius of curvature and center of curvature to the parabola \((x, y) = (at^2, 2at)\), where \(a\) is a constant.
(b) Find the radius of curvature at a general point \((x, y)\) on the graph of \(y = 2x + 3\).
(c) Find the point of maximum curvature on the parabola \(y = x^2\).

Problem 2 (Classes 22 (15 pts: 10,5))
(a) Define the cycloid and derive parametric equations for it.
(b) Compute the arc length of one arch of the cycloid.

Problem 3 (Class 20 (10 pts))
Find the center of the unique circle through the three points \((1, 0, 0), (0, 2, 0)\) and \((0, 0, 1)\).
Problem 4 (Class 24, 10 pts: 5,5)
Place a unit cube in the corner of the first octant with edges along the axes. For this problem consider the front face diagonal containing \((1, 0, 1)\) and the right face diagonal containing \((0, 1, 0)\). These two lines are skew; the problem is to find the length and position of the shortest line segment joining them.

(a) Draw a picture and write parametric equations for the two lines containing these two diagonals. For clarity, use different variables, \(t\) and \(u\), as the parameters for the two lines.

(b) Let \(w(t, u)\) be the square of the distance between a point on \(A\) on the front diagonal and a point \(B\) on the side diagonal. Find the (unique) values of \(t\) and \(u\) where both \(\frac{\partial w}{\partial t} = 0\) and \(\frac{\partial w}{\partial u} = 0\). (This is called a critical point. It gives the values of \(t\) and \(u\) which give the endpoints of the shortest segment connecting the two diagonals.)

We will revisit this problem in the next problem set.

Problem 5 (Class 22: 20 pts: 10,5,5) A hockey puck of radius 1 slides along the ice at a speed \(10\sqrt{2}\) in the direction of the vector \((1, 1)\). As it slides, it spins in a counterclockwise direction at 2 revolutions per unit time. At time \(t = 0\), the puck’s center is at the origin \((0, 0)\).

(a) Find the parametric equations for the trajectory of the point \(P\) at the edge of the puck initially at \((1, 0)\).

(b) Find the velocity \(v\) of the point \(P\).

(c) What is the minimum speed of the point \(P\), and what is the direction of the velocity at the corresponding time?

Problem 6 (Class 22: 16 pts: 4,4,4,4) Consider the helical trajectory with position vector
\[
r = \sin(4t)i + \cos(4t)j + 3t k.
\]

(a) Calculate the velocity vector \(v\) and the unit tangent vector \(T\).

(b) Calculate the speed \(ds/dt\) and the arclength traced out between the points at \(t = 0\) and \(t = 2\pi\).

(c) Calculate the curvature \(\kappa\).

(d) Show that the curve makes a constant angle with the \(k\)-direction.

Problem 7 (Class 24: 5 pts) Show that \(z = \tan^{-1}(y/x)\) satisfies \(z_{xx} + z_{yy} = 0\).

End of pset 7