16 Eigenvalues, diagonalization, decoupling

This note covers topics that will take us several classes to get through. We will look almost exclusively at $2 \times 2$ matrices. These have almost all the features of bigger square matrices and they are computationally easy.

16.1 Etymology:

This is from a Wikipedia discussion page: The word *eigen* in German or Dutch translates as 'inherent', 'characteristic', 'private'. So *eigenvector* of a matrix is characteristic or inherent to the matrix. The word eigen is also translated as 'own' with the same sense as the meanings above. That is the eigenvector of a matrix is the matrix’s 'own vector'.

In English you sometimes see eigenvalues called special or characteristic values.

16.2 Definition

For a square matrix $M$ an *eigenvalue* is a number (scalar) $\lambda$ that satisfies the equation

$$M \mathbf{v} = \lambda \mathbf{v} \text{ for some non-zero vector } \mathbf{v}.$$  

(1)

The vector $\mathbf{v}$ is called a *non-zero eigenvector corresponding to* $\lambda$. We will call Equation 16.1 the *eigenvector equation*.

**Comments:**

1. Using the symbol $\lambda$ for the eigenvalue is a fairly common practice when looking at generic matrices. If the eigenvalue has a physical interpretation we’ll often use a corresponding letter. For example, in population matrices the eigenvalues are growth rates, so we’ll often denote them using $r$ or $k$.

2. Eigenvectors are not unique. That is, if $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda$ then so any multiple of $\mathbf{v}$. Indeed, the set of all eigenvectors with eigenvalue $\lambda$ is clearly a vector space. (You should show this!)

16.3 Why eigenvectors are special

**Example 16.1.** Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

We will explore how $A$ transforms vectors and what makes an eigenvector special. We will see that $A$ scales and rotates most vectors, but only scales eigenvectors. That is, eigenvectors lie on lines that are unmoved by $A$.

Take $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ \Rightarrow $A\mathbf{u}_1 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$; Take $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ \Rightarrow $A\mathbf{u}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

We see that $A$ scales and turns most vectors.
Now take \( \mathbf{v}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \) \( \Rightarrow \mathbf{A} \mathbf{v}_1 = \begin{bmatrix} 35 \\ 7 \end{bmatrix} = 7 \mathbf{v}_1 \). This shows that \( \mathbf{v}_1 \) is an eigenvector with eigenvalue 7. The eigenvector is special since \( \mathbf{A} \) just scales it by 7.

Likewise, \( \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) then \( \mathbf{A} \mathbf{v}_2 = \mathbf{v}_2 \). So, \( \mathbf{v}_2 \) is an eigenvector with eigenvalue 1. The eigenvector \( \mathbf{v}_2 \) is really special, it is unmoved by \( \mathbf{A} \).

The following example shows how knowing eigenvalues and eigenvectors simplifies calculations with a matrix. In fact, you don’t even need the matrix once you know all of its eigenvalues and eigenvectors.

**Example 16.2.** Suppose \( \mathbf{A} \) is a \( 2 \times 2 \) matrix that has eigenvectors \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) with eigenvalues 2 and 4 respectively.

(a) Compute \( \mathbf{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

**answer:** Since \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) is an eigenvector this follows directly from the definition of eigenvectors:

\[
\mathbf{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.
\]

(b) Compute \( \mathbf{A} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \).

**answer:** This uses the definition of eigenvector plus linearity:

\[
\mathbf{A} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = \mathbf{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \mathbf{A} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \end{bmatrix}.
\]

(c) Compute \( \mathbf{A} \left( 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \).

**answer:** Again this uses the definition of eigenvector plus linearity:

\[
\mathbf{A} \left( 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = 3 \mathbf{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 5 \mathbf{A} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 20 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 26 \\ 72 \end{bmatrix}.
\]
(d) Compute $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

**Answer:** We first decompose $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into eigenvectors:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$  

Now we can once again use the definition of eigenvector plus linearity:

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = A \begin{bmatrix} 1 \\ 3 \end{bmatrix} - A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$  

**Example 16.3.** Any rotation in three dimensions is around some axis. The vector along this axis is fixed by the rotation, i.e. it is an eigenvector with eigenvalue 1.

### 16.4 Computational approach

In order to find the eigenvectors and eigenvalues we recall the following basic fact about square matrices.

**Fact:** The equation $Mv = 0$ is satisfied by a non-zero vector $v$ if and only if $\det(M) = 0$. (Said differently, the null space of $M$ is non-trivial exactly when $\det(M) = 0$.)

First we manipulate the eigenvalue equation 1 so that finding eigenvectors becomes finding a null vectors:

$$Av = \lambda v \iff Av = \lambda Iv \iff Av - \lambda Iv = 0 \iff (A - \lambda I)v = 0.$$  

Now we can apply our 'fact' to the last equation above to conclude that $\lambda$ is an eigenvalue if and only if

$$\det(A - \lambda I) = 0 \quad (2)$$  

(Recall that an definition of eigenvalue requires $v$ to be non-zero.)

We call Equation 2 the characteristic equation. (Eigenvalues are sometimes called characteristic values.) It allows us to find the eigenvalues and eigenvectors separately in a two step process.

**Notation:** For simplicity we will use the notation $|A| = \det(A)$.

**Eigenspace:** For an eigenvalue $\lambda$ the set of all eigenvectors is none other than the nullspace of $A - \lambda I$. That is, it is a vector subspace which we call the eigenspace of $\lambda$.

**Example 16.4.** Find the eigenvalues of the matrix $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$. For each eigenvalue find a basis of the corresponding eigenspace.

**Answer:** Step 1: Find the eigenvalues $\lambda$:

$$|A - \lambda I| = 0 \quad (\text{characteristic equation})$$

$$\Rightarrow \left| \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0 \Rightarrow \left| \begin{bmatrix} 6 - \lambda & 5 \\ 1 & 2 - \lambda \end{bmatrix} \right| = 0.$$
Step 2: Find basis eigenvectors $v$: We have $(A - \lambda I) v = 0$, i.e. we want Null$(A - \lambda I)$.

$\lambda_1 = 7$: $A - \lambda I = \begin{bmatrix} -1 & 5 \\ 1 & -5 \end{bmatrix}$. This has RREF $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The nullspace is 1 dimensional, a basis is $v_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

$\lambda_2 = 1$: $A - \lambda I = \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}$. This has RREF $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The nullspace is 1 dimensional, a basis is $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Remember, any scalar multiple of these eigenvectors is also an eigenvector with the same eigenvalue.

**Trick.** In the $2 \times 2$ case we don’t have to write out the RREF to find the eigenvector. Notice that the entries in our eigenvectors come from the entries in one row of the matrix. You use the right entry of the row and minus the left entry. If you think about this a moment you’ll see that it is always the case.

**Matlab:** In Matlab the function `eig(A)` returns the eigenvectors and eigenvalues of a matrix.

**Complex eigenvalues**
If the eigenvalues are complex then the eigenvectors are complex. Otherwise there is no difference in the algebra.

**Example 16.5.** Find the eigenvalues and basic eigenvectors of the matrix $A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$.

**answer:** Step 1: Find the eigenvalues $\lambda$: $|A - \lambda I| = 0$ (characteristic equation)

$\Rightarrow \begin{vmatrix} 3 - \lambda & 4 \\ -4 & 3 - \lambda \end{vmatrix} = 0 \Rightarrow (3 - \lambda)^2 + 16 = 0 \Rightarrow \lambda = 3 \pm 4i.$

Step 2, Find the eigenvectors $v$: $(A - \lambda I)v = 0$.

$\lambda_1 = 3 + 4i$: $\begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix} v_1 = 0$. Take $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

$\lambda_2 = 3 - 4i$: $\begin{bmatrix} 4i & 4i \\ -4 & 4i \end{bmatrix} v_2 = 0$. Take $v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Notice that the eigenvalues and eigenvectors come in complex conjugate pairs. Knowing this there is no need to do a computation to find the second member of each pair.

**Example 16.6.** Find the eigenvalues and basic eigenvectors of the matrix $A = \begin{bmatrix} 1 & -4 \\ 5 & 5 \end{bmatrix}$.

**answer:** Step 1: find $\lambda$ (eigenvalues): $|A - \lambda I| = 0$ (characteristic equation)

$\Rightarrow \begin{vmatrix} 1 - \lambda & -4 \\ 5 & 5 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 6\lambda + 25 = 0 \Rightarrow \lambda = 3 \pm 4i.$
Step 2, find \( v \) (eigenvectors): \((A - \lambda I)v = 0\).

\[
\lambda_1 = 3 + 4i: \begin{bmatrix} -2 - 4i & -4 \\ 5 & 2 - 4i \end{bmatrix} v_1 = 0. \quad \text{Take } v_1 = \begin{bmatrix} 4 \\ -2 - 4i \end{bmatrix}.
\]

\[
\lambda_2 = 3 - 4i: \quad \text{take } v_2 = v_1 = \begin{bmatrix} 4 \\ -2 + 4i \end{bmatrix}.
\]

**Repeated eigenvalues**

When a matrix has repeated eigenvalues the eigenvectors are not as well behaved as when the eigenvalues are distinct. There are two main examples

**Example 16.7. Defective case**

Find the eigenvalues and basic eigenvectors of the matrix \( A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \).

**answer:** Step 1: Find the eigenvalues \( \lambda \): \(|A - \lambda I| = 0\) (characteristic equation)

\[
|A - \lambda I| = 0 \Rightarrow \left| \begin{array}{cc} 3 - \lambda & 1 \\ 0 & 3 - \lambda \end{array} \right| = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3, 3.
\]

Step 2, Find the eigenvectors \( v \): \((A - \lambda I)v = 0\).

Note: because the eigenvalue is repeated, we hope to find a two dimensional eigenspace.

\[
\lambda_1 = 3: \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v_1 = 0. \quad \text{This is already in RREF. It has one free variable, so the nullspace is 1 dimensional. We can take a basis vector: } v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

There are no other eigenvalues and we can’t find another independent eigenvector with eigenvalue 3. We have two eigenvalues but only one independent eigenvector, so we call this case defective or incomplete.

In terms of eigenspaces and dimensions we say the repeated eigenvalue \( \lambda = 3 \) has a one dimensional eigenspace. It is defective because we want two dimensions from the double root.

**Example 16.8. Complete case** Find the eigenvalues and basic eigenvectors of the matrix \( A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \).

**answer:** Step 1: Find the eigenvalues \( \lambda \): \(|A - \lambda I| = 0\) (characteristic equation)

\[
|A - \lambda I| = 0 \Rightarrow \left| \begin{array}{cc} 3 - \lambda & 0 \\ 0 & 3 - \lambda \end{array} \right| = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3, 3.
\]

Step 2, Find the eigenvectors \( v \): \((A - \lambda I)v = 0\).

\[
\lambda_1 = 3: \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} v_1 = 0. \quad \text{This equation shows that every vector in } \mathbb{R}^2 \text{ is an eigenvector. That is, the eigenvalue } \lambda = 3 \text{ has a two dimensional eigenspace. We can pick any two independent vectors as a basis, e.g. } v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad \text{(These are the simplest choices, but any two independent vectors}
\]
would work!) Because we have as many independent eigenvectors as eigenvalues we call this case **complete**.

### 16.5 Diagonal matrices

In this section we will see how easy it is to work with diagonal matrices. In later sections we will see how working with eigenvalues and eigenvectors of a matrix is like turning it into a diagonal matrix.

**Example 16.9.** Consider the diagonal matrix

\[ B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \]

Convince yourself that \( B \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2u \\ 3v \end{bmatrix} \). That is \( B \) scales the first coordinate by 2 and the second coordinate by 3.

We can write this as

\[ B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

This is exactly the definition of eigenvectors. That is, \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) are eigenvectors with eigenvalues 2 and 3 respectively. That is, for a diagonal matrix the diagonal entries are the eigenvalues and the eigenvectors all point along the coordinate axes.

### 16.6 Diagonal matrices and uncoupled algebraic systems

**Example 16.10.** An uncoupled algebraic system Consider the system

\[
\begin{align*}
7u & = 1 \\
v & = 3
\end{align*}
\]

This barely deserves to be called a system. The variables \( u \) and \( v \) are uncoupled and we solve the system by finding each variable separately: \( u = 1/7, \ v = 3 \).

**Example 16.11.** Now consider the system

\[
\begin{align*}
6x + 5y & = 2 \\
x + 2y & = 4.
\end{align*}
\]

In matrix form this is

\[ \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \] (3)

The matrix \( A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \) is the same matrix as in Examples 16.1 and 16.4 above. In this system the variables \( x \) and \( y \) are coupled. We will explain the logic of decoupling later. For this example, we will decouple the equations using some magical choices involving eigenvectors.

The examples above showed that the eigenvalues of \( A \) are 7 and 1 and the eigenvectors are \( \begin{bmatrix} 5 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). We write our vectors in terms of the eigenvectors by making the change of
variables
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = u \begin{bmatrix}
5 \\
1
\end{bmatrix} + v \begin{bmatrix}
-1 \\
1
\end{bmatrix} \iff x = 5u - v; \quad y = u + v.
\]

We also note that \[
\begin{bmatrix}
2 \\
4
\end{bmatrix} = \begin{bmatrix}
5 \\
1
\end{bmatrix} + 3 \begin{bmatrix}
-1 \\
1
\end{bmatrix}.
\]
Converting \(x\) and \(y\) to \(u\) and \(v\) we get

\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \left( u \begin{bmatrix}
5 \\
1
\end{bmatrix} + v \begin{bmatrix}
-1 \\
1
\end{bmatrix} \right) = 7u \begin{bmatrix}
5 \\
1
\end{bmatrix} + v \begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]

Thus
\[
\begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
2 \\
4
\end{bmatrix} \iff 7u \begin{bmatrix}
5 \\
1
\end{bmatrix} + v \begin{bmatrix}
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
5 \\
1
\end{bmatrix} + 3 \begin{bmatrix}
-1 \\
1
\end{bmatrix}
\]

It is easy to see that the last system is the same as the equations

\[7u = 1 \quad v = 3.\]

In \(u, v\) coordinates the system is diagonal and easy to solve.

### 16.7 Introduction to matrix methods for solving systems of DE’s

In this section we will solve linear, homogeneous, constant coefficient systems of differential equations using the matrix methods we have developed. For now we will just consider matrices with real, distinct eigenvalues. In the next topic we will look at complex and repeated eigenvalues.

As with constant coefficient DE’s we will use the method of optimism to discover a systematic technique for solving systems of DE’s. We start by giving the general 2 \times 2 linear, homogeneous, constant coefficient system of DE’s. It has the form

\[
x' = ax + by \\
y' = cx + dy.
\]

(4)

Here \(a,b,c,d\) are constants and \(x(t), y(t)\) are the unknown functions we need to solve for.

There are a number of important things to note.

1. We can write Equation 4 in matrix form

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} \iff \mathbf{x}' = A\mathbf{x}
\]

(5)

where \(A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\) and \(\mathbf{x} = \begin{bmatrix}
x \\
y
\end{bmatrix}\).

2. The system is homogeneous. You can see this by taking Equation 4 and putting all the \(x\) and \(y\) on the left side so that the right side becomes all zeros.

3. The system is linear. You should be able to check directly that a linear combination of solutions to Equation 5 is also a solution.

We illustrate the method of optimism for solving Equation 5 in an example.

**Example 16.12.** Solve the linear, homogeneous, constant coefficient system

\[
\mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{bmatrix}
x \\
y
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}
6 & 5 \\
1 & 2
\end{bmatrix}.
\]
answer: Using the method of optimism we try a solution

\[ x = e^{\lambda t}v, \]

where \( \lambda \) is a constant and \( v \) is a constant vector. Substituting the trial solution into both sides of the DE we get

\[ \lambda e^{\lambda t}v = e^{\lambda t}Av \Leftrightarrow Av = \lambda v. \]

This is none other than the eigenvalue/eigenvector equation. So solving the system amounts to finding eigenvalues and eigenvectors. From our previous examples we know the eigenvalues and eigenvectors of \( A \). We get two solutions.

\[ x_1 = e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad x_2 = e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}. \]

The general solution is the span of these solutions:

\[ x = c_1 x_1 + c_2 x_2 = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}. \]

The solutions \( x_1 \) and \( x_2 \) are called modal solutions.

Now that we know where the method of optimism leads we can do a second example starting directly with finding eigenvalues and eigenvectors

**Example 16.13.** Find the general solution to the system

\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

answer: First find eigenvalues and basic eigenvectors.

Characteristic equation: \( \begin{vmatrix} 3 - \lambda & 4 \\ 1 & 3 - \lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 1, 5. \)

Eigenvalues: solve \( (A - \lambda I)v = 0 \)

\( \lambda = 1: \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v = 0. \) Take \( v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \)

\( \lambda = 5: \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v = 0. \) Take \( v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \)

We have two modal solutions: \( x_1 = e^{\lambda t}v_1 \) and \( x_2 = e^{5\lambda t}v_2. \)

The general solution is \( x = c_1 x_1 + c_2 x_2 = c_1 e^{\lambda t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5\lambda t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \)

**16.8 Decoupling systems of DE’s**

**Example 16.14.** (An uncoupled system) Consider the system

\[ \begin{align*}
    u'(t) &= 7u(t) \\
    v'(t) &= v(t)
\end{align*} \]
Since $u$ and $v$ don’t have any effect on each other we say the $u$ and $v$ are uncoupled. It’s easy to see the solution to this system is

$$u(t) = c_1 e^{7t}$$
$$v(t) = c_2 e^t$$

In matrix form we have

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$  

The coefficient matrix has eigenvalues are 7 and 1, with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The general solution to the system of DEs is

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^{7t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

We see an uncoupled system has a diagonal coefficient matrix and the eigenvectors are the standard basis vectors. All in all, it’s simple and easy to work with.

The following example shows how to decouple a coupled system.

**Example 16.15.** Consider once again the system from Example 16.12

$$\begin{align*}
  x' &= 6x + 5y \\
  y' &= x + 2y.
\end{align*}$$  

$\iff x' = Ax$, where $x = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ \hspace{1cm} (6)

In this system the variables $x$ and $y$ are coupled. Make a change of variable that converts this to a decoupled system.

**answer:** From Example 16.12 we know the eigenvalues are 7 and 1, the eigenvectors are $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and the general solution is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Notice that $c_1 e^{7t}$ and $c_2 e^t$ in the above solution are just $u$ and $v$ from the previous example. So we can write

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = u(t) \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v(t) \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \hspace{1cm} (7)$$

This is a change of variables. Notice that $u$ and $v$ are the coefficients in the decomposition of $\begin{bmatrix} x \\ y \end{bmatrix}$ into eigenvectors. Also, since $x$ and $y$ depend on $t$ so do $u$ and $v$.

Let’s rewrite the system in Equation 6 in terms of $u$, $v$. Using Equation 7 we get

$$x' = \begin{bmatrix} x' \\ y' \end{bmatrix} = u' \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v' \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} x \\ y \end{bmatrix} = A \left( u \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = 7u \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$  

The last equality follows because $\begin{bmatrix} 5 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$ are eigenvectors of $A$.

Equating the two sides we get

$$u' \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v' \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 7u \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$
Comparing the coefficients of the eigenvectors we get
\[ \begin{align*}
    u' &= 7u \\
    v' &= v
\end{align*} \iff \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \]

That is, in terms of \( u \) and \( v \) the system is uncoupled. Note that the eigenvalues of \( A \) are precisely the diagonal entries of the uncoupled system.

### 16.8.1 Decoupling in general

To end this section we’ll redo the previous example to show a systematic method for decoupling a system.

**Example 16.16.** Decouple the system in Example 16.15.

**answer:** Let \( S = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \) = matrix with eigenvectors of \( A \) as columns.

Let \( \Lambda = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \) = diagonal matrix with the eigenvalues of \( A \) as diagonal entries.

In the example we made a change of variable using \( \begin{bmatrix} x \\ y \end{bmatrix} = u \begin{bmatrix} 5 \\ 1 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). In matrix form this is just \( \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = S \begin{bmatrix} u \\ v \end{bmatrix} \).

Inverting we have the change of variables \( \begin{bmatrix} u \\ v \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \). In Example 16.15 we found that this gives
\[ \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \Lambda \begin{bmatrix} u \\ v \end{bmatrix}. \]

Since \( u \) and \( v \) are uncoupled the change of variables \( u = S^{-1}x \) is called decoupling the system.

**Abstract form.** We finish by noting in symbolic form our systematic method of decoupling.

Suppose we have the system \( \mathbf{x}' = A\mathbf{x} \). If \( S \) is a matrix with the eigenvectors of \( A \) as columns and \( \Lambda \) is the diagonal matrix with the corresponding eigenvalues as diagonal entries. Then the change of variables \( \mathbf{u} = S^{-1}\mathbf{x} \) changes the coupled system \( \mathbf{x}' = A\mathbf{x} \) into an uncoupled system \( \mathbf{u}' = \Lambda\mathbf{u} \).

**Proof.** The key is diagonalization which is discussed in the next section. This says that \( A = SAS^{-1} \). Thus, if \( \mathbf{x} = Su \) we have
\[ \mathbf{x}' = A\mathbf{x} \iff Su' = ASu \iff Su' = SAS^{-1}Su = S\Lambda u \iff u' = \Lambda u \]

### 16.9 Diagonalization

Decoupling by changing to coordinates based on eigenvectors can be organized into what is called the diagonalization of a matrix.

**Theorem.** Diagonalization theorem. Suppose the \( n \times n \) matrix \( A \) has \( n \) independent eigenvectors. Then, we can write
\[ A = SAS^{-1}, \]
where $S$ is a matrix whose columns are the $n$ independent eigenvalues and $A$ is the diagonal matrix whose diagonal entries are the corresponding eigenvalues.

The proof is below. We illustrate this first with our standard example.

**Example 16.17.** We know the matrix $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ has eigenvalues 7 and 1 with corresponding eigenvectors $v_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}^T$.

We put the eigenvectors as the columns of a matrix $S$ and the eigenvalues as the entries of a diagonal matrix $\Lambda$.

$$S = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

The diagonalization theorem says that

$$A = SAS^{-1} = \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ -1/6 & 5/6 \end{bmatrix}.$$ 

This is called the diagonalization of $A$. Note the form: a diagonal matrix $\Lambda$ surrounded by $S$ and $S^{-1}$.

**Proof of the diagonalization theorem.** We will do this for the matrix in the example above. It will be clear that this proof carries over to any $n \times n$ matrix with $n$ independent eigenvectors. We could check directly that the diagonalization equation holds. Instead we will give an argument based on matrix multiplication. Because the eigenvectors form a basis of $\mathbb{R}^2$, superposition tells us that it will be enough to show that $A$ and $SAS^{-1}$ have the same effect on the eigenvectors $v_1$ and $v_2$.

Recall that multiplying a matrix times a column vector results in a linear combination of the columns. In our case

$$S \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [v_1 \ v_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = v_1, \quad \text{likewise} \quad S \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_2.$$ 

Inverting we have

$$S^{-1}v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad S^{-1}v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

Now we can check that $SAS^{-1}v_1 = Av_1$ and $SAS^{-1}v_2 = Av_2$:

$$SAS^{-1}v_1 = SA \begin{bmatrix} 1 \\ 0 \end{bmatrix} = S \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = S \begin{bmatrix} 7 \\ 0 \end{bmatrix} = 7v_1 = Av_1$$

The last equality follows because $v_1$ is an eigenvector of $A$ with eigenvalue 7. The equation of $v_2$ is just the same. Since the two matrices have the same effect when multiplied by a basis $v_1, v_2$ of $\mathbb{R}^2$ they are the same matrix. QED

The example shows the general method for diagonalizing an $n \times n$ matrix $A$. The steps are:

1. Find the eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$.
2. Make the matrix of eigenvectors $S = [v_1 \ v_2 \ \cdots \ v_n]$
3. Make the diagonal matrix of eigenvalues \( \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \)

The diagonalization is: \( A = SAS^{-1} \).

**Note:** Diagonalization requires that \( A \) have a full complement of eigenvectors. If it is defective it can’t be diagonalized.

We have the following important formula

\[ \det(A) = \text{product of its eigenvalues}. \]

This follows easily from the diagonalization formula

\[ \det(A) = \det(SAS^{-1}) = \det(S) \det(\Lambda) \det(S^{-1}) = \det(\Lambda) = \text{product of diagonal entries}. \]

### 16.10 Symmetric matrices

This section is optional. We won’t ask about it on psets or tests. The first example in this section is a nice exercise in thinking about matrix multiplication as a way to transform vectors.

**Example 16.18. Geometry of symmetric matrices.** This is a fairly complex example showing how we can use the diagonal matrix \( \Lambda = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \) and the rotation matrix \( R = \begin{bmatrix} \cos \theta & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \) to convert a circle to an ellipse as shown in the figures below.

To do this we think of matrix multiplication as a linear transformation. The diagonal matrix \( \Lambda \) transforms the circle by scaling the \( x \) and \( y \) directions by \( a \) and \( b \) respectively. This creates the ellipse in (b) which is oriented with the axes. The rotation matrix \( R \) then rotates this ellipse to the general ellipse in (c).
In coordinates $R\Lambda$ maps the unit circle $u^2 + v^2 = 1$ to the ellipse shown in (c). That is,

$$R\Lambda \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} u \\ v \end{bmatrix} = \Lambda^{-1} R^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

**Example 16.19. Spectral theorem.** The previous example transforms the unit circle in $wv$-coordinates into an ellipse in $xy$-coordinates. In terms of inner products and transposes this becomes

$$1 = \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle = \left\langle \Lambda^{-1} R^{-1} \begin{bmatrix} x \\ y \end{bmatrix}, \Lambda^{-1} R^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \begin{bmatrix} x \\ y \end{bmatrix}^T (\Lambda^{-1} R^{-1})^T \Lambda^{-1} R^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^T R \Lambda^{-2} R^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

The last equality uses the facts that for a rotation matrix $R^T = R^{-1}$ and for a diagonal matrix $\Lambda^T = \Lambda$.

Call the matrix occurring in the last two lines above

$$A = R\Lambda^{-2} R^{-1} = (\Lambda^{-1} R^{-1})^T \Lambda^{-1} R^{-1}.$$

We then have the equation of the ellipse is

$$1 = \begin{bmatrix} x \\ y \end{bmatrix}^T A \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix $A$ has the following properties.
1. It is symmetric
2. Its eigenvalues are $a^{-2}$ and $b^{-2}$
3. Its eigenvectors are the vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ along the axes of the ellipse (see figure (c) above).
4. Its eigenvectors are orthogonal.

**Proof.**
1. This is clear from the formula $A = B^T B$ where $B = \Lambda^{-1} R^{-1}$.
2. This is clear from the diagonalization $A = R \Lambda^{-2} R^{-1}$. (Remember the eigenvalues are in the diagonal matrix $\Lambda^{-2}$.
3. We need to show that $A$ transforms $\overrightarrow{v_1}$ to a multiple of itself. This also follows by considering the action of each term in the diagonalization in turn (see the figures): $R^{-1}$ moves $\overrightarrow{v_1}$ to the $x$-axis; then $\Lambda^{-2}$ scales the $x$-axis by $a^{-2}$; and finally $R$ rotates the $x$-axis back the line along $\overrightarrow{v_1}$. Using symbols

$$A \overrightarrow{v_1} = R \Lambda^{-2} R^{-1} \overrightarrow{v_1} = R \Lambda^{-2} a \overrightarrow{i} = R(a^{-2} a \overrightarrow{i}) = a^{-2} \overrightarrow{v_1}$$

The properties of $A$ are general properties of symmetric matrices.

**Spectral theorem.** A symmetric matrix $A$ has the following properties.
1. It has real eigenvalues.
2. Its eigenvectors are mutually orthogonal.

Because of the connection to the axes of ellipses this is also called the principal axis theorem.