8 Applications, stability

8.1 Goals
1. Know the meaning of the term 'linear time invariance'.
2. Be able to apply linear time invariance to solve equations with input shifted in time.
3. Know the definitions of mathematical and physical stability
4. Be able to determine if a given 1st, 2nd or 3rd order system is stable.

8.2 Time invariance

Constant coefficient differential equations have the property of time invariance. Physically this means that the system responds the same way to an input no matter when the input is started. Mathematically we write this as

**Definition.** Time invariance of a constant coefficient system is the property that if $x_p(t)$ satisfies $P(D)x = f(t)$ then $x_p(t - t_0)$ satisfies $P(D)x = f(t - t_0)$.

**Example 8.1.** We know that $x' + 3x = e^{-t}$ has solution $x_1(t) = e^{-t}/2$. Time invariance says that $x' + 3x = e^{-(t-3)}$ has solution $x_2(t) = x_1(t-3) = e^{-(t-3)}/2$. This the figure below illustrates that shifting the input in time simply shifts the output in time.

![Diagram showing time invariance](image)

Physically this has to be the case – an exponential decay system doesn’t care what time it gets started.

8.3 Mathematical stability

We introduce the idea of stability with an example that shows how negative exponents imply that initial conditions do not affect the long-term behavior of a system.

**Example 8.2.** Solve the initial value problem (IVP) and describe the long-term behavior of the system.

\[
x'' + 2x' + 3x = \cos(2t), \quad x(0) = 2, \, x'(0) = 3
\]
**answer:** As usual we find a particular solution and the general homogeneous solution.

*Particular solution.* In order to apply the Sinusoidal Response Formula (SRF) we need to use the characteristic polynomial:

\[ P(2i) = -4 + 4i + 3 = -1 + 4i = \sqrt{17}e^{i\phi}, \]

where \( \phi = \text{Arg}(P(2i)) = \text{Arg}(-1 + 4i) = \tan^{-1}(-4) \) in the 2nd quadrant. Now the SRF gives \( x_p(t) = \frac{\cos(2t - \phi)}{\sqrt{17}} \). (You should remind yourself how to derive the SRF using complexification and the ERF)

*Homogeneous solution.* The characteristic equation is \( r^2 + 2r + 3 = 0. \)

Roots: \( r = -1 \pm \sqrt{2}i. \)

So, \( x_h(t) = c_1e^{-t} \cos(\sqrt{2}t) + c_2e^{-t} \sin(\sqrt{2}t) \)

*General solution.*

\[ x(t) = x_p(t) + x_h(t) = x_p(t) + c_1e^{-t} \cos(\sqrt{2}t) + c_2e^{-t} \sin(\sqrt{2}t). \]

We won’t go through the arithmetic, but using the initial conditions we can compute \( c_1 = \frac{35}{17} \) and \( c_2 = \frac{44\sqrt{2}}{17}. \)

The question also asks what happens to the system in the long-term, i.e. at \( t \to \infty. \) Looking at the general solution we see that \( x_h(t) \) has negative exponents, so it goes to 0 in the long-term. This means that in the long-term the system settles down to the periodic solution \( x_p(t) = \cos(2t - \phi)/\sqrt{17}. \)

This is a key point: the initial conditions determine the coefficients \( c_1 \) and \( c_2. \) but in the long-term those terms go to 0, i.e. the initial conditions don’t affect the long-term behavior of the system.

This leads to our definition of stability and several equivalent ways of describing it.

**Definition.** Mathematical stability means the long-term behavior doesn’t depend (significantly) on initial conditions.

**Linear Systems.** The system \( Ly = f \) is stable if for any IC \( y_h(t) \to 0 \) as \( t \to \infty. \) In this case, \( y_h \) is called the transient.

**Linear CC Systems.** The system \( P(D)y = f \) is stable if all the characteristic roots have negative real part.

For linear systems stability was determined by the homogeneous solution. That is, **Stability is about the system not the input.**

**Example 8.3.** \( x' + 2x = f(t) \) is stable because \( x_h(t) = ce^{-2t} \to 0. \)

**Example 8.4.** A constant coefficient system with roots \( -2 \pm 3i, -3 \) is stable.

**Example 8.5.** A constant coefficient system with roots \( -2, -3, 4 \) is unstable.

**Example 8.6.** \( P(D)y = y'' + 8y' + 7y = f(t) \) has characteristic roots -7, -1. These are negative so the system is stable.

**Example 8.7.** \( P(D)y = y'' - 6y' + 25y = f \) has characteristic roots \( 3 \pm 4i. \) The real parts of these roots are positive, so the system is not stable.
8.4 Stability criteria for linear CC systems

1. Stability ⇔ for any IC $y_h \to 0$ as $t \to \infty$.

2. Stability ⇔ all characteristic roots have negative real part.

3. Stability ⇔ all solutions to the homogeneous equation $P(D)y = 0$ go asymptotically to the homogeneous equilibrium solution $y(t) = 0$.

4. For a first order system $P(D)y = y' + ky = f$: root $= -k \Rightarrow$ stability $\iff k > 0$.

5. For a second order system $P(D)y = y'' + ay' + by = f$: stability $\iff a > 0$ and $b > 0$ (easy to prove).

6. For a third order system $P(D)y = y''' + ay'' + by' + cy = f$: stability $\iff a, b, c > 0$ and $ab > c$ (harder to prove).

   This shows that third order systems with positive coefficients aren’t necessarily stable.

   Example: An unstable system with positive coefficients
   
   $$(r + 5)(r - 1 - 100i)(r - 1 + 100i) = r^3 + 3r^2 + 96r + 505.$$  

7. For higher order systems there is the Routh-Hurwitz stability criteria, which is described in the supplementary notes: [http://math.mit.edu/~jorloff/suppnotes/suppnotes03/s.pdf](http://math.mit.edu/~jorloff/suppnotes/suppnotes03/s.pdf). This is somewhat complicated, but it allows us to determine stability from the coefficients of a system. That is, it does not require finding the roots!

8.5 Physical stability

Definition. Physical stability. An unforced physical system with a single equilibrium is called stable if for any initial conditions it returns to the equilibrium.

Later in the course we will expand on the notion of stability for systems with multiple equilibria. The next example shows how physical and mathematical stability are related.

Example 8.8. Damped-spring-mass system: Physical stability matches mathematical stability. The equilibrium solution is $x(t) = 0$. The system is modeled by $x' + bx' + kx = 0$ and since the roots have negative real part $x(t) \to 0$ no matter what the initial conditions.

Note: The previous section on stability criteria show that 2nd order physical systems, like springs and LRC circuits are always stable. This is not true of 3rd (and higher) order physical systems.