18.03 Practice Questions – Exam 1, Fall 2017

(This will take considerably longer than 1 hour; the actual exam will be shorter.)

Remember: practice exams can’t include all possible problem types. You should also look at the class notes and psets to prepare.

Problem 1
Solve the following DE. \((\cos(x)) \frac{dy}{dx} + (\sin(x) \cos(y)) = 0\)

Problem 2
(a) Compute the following real function of \(x\): \(\text{Im}\left(\frac{e^{(3+2i)x}}{3 + 2i}\right)\)
   (where as usual \(\text{Im}\) denotes the imaginary part of a complex number).

(b) Use the result of part(a) to compute the integral \(\int e^{3x} \sin(2x) \, dx\) using the complex exponential.

Problem 3
(a) Find the general real-valued solution to the DE \(y'' + 4y' + 13y = 0\) and the solution satisfying the initial conditions (IC) \(y(0) = 1, y'(0) = 0\).

(b) For what values of \(b\) will all the (non-zero) solutions to \(y'' + by' + 13y = 0\) display oscillatory behavior?

(c) For these oscillatory solutions, in theory how many times does each solution cross the positive \(t\)-axis (or whatever letter you picked for the independent variable – now but think of it as representing time.) If this DE is modeling some real-world situation what actually happens happens to the quantity \(y = y(t)\) in the long run?

Problem 4
A salt solution of strength 2 grams per liter is flowing into a 50 liter tank, and the solution in the tank is being pumped out, both at the (same) time-varying rate of \(\frac{1}{1+t}\) liters per minute.

(a) With the usual instantaneous-mixing assumptions: derive the DE for the rate of change of the amount of salt \(x = x(t)\) (in grams) in the tank with respect to time (in minutes).

(b) Solve this DE explicitly for \(x = x(t)\) with the IC \(x(0) = 0\) (i.e. starting off with pure water in the tank).

(c) Using both (i) the solution \(x = x(t)\) found in part (b) and (ii) an argument from ‘first principles’ (i.e. directly from the physical situation described here), answer the following question: what happens to the amount of salt in the tank in the long-run over time? In particular, does it approach a final limiting value? If so, what is this value?
   (The idea is to do both (i) and (ii) and show they give the same prediction for the long-term behavior.)

Problem 5
Given the DE \(y'' + 4y' + 5y = 8 \cos(2t)\):

(a) Find the general solution to the DE.
(b) What is the periodic solution? Give your answer in amplitude-phase form.

(c) Show that, no matter what the initial conditions, the system always settles down to the periodic solution.

Problem 6
Let \( P(D) = D^2 + bD + I \) where \( D = \frac{d}{dt} \) and \( b > 0 \).

(a) For what range of the values of \( b \) will the solutions to \( P(D)y = 0 \) exhibit oscillatory behavior?

(b) Describe the different types of graphs, \( y = y(t) \), one gets for values of \( b > 0 \).

(c) For \( b = 1 \), solve the DE \( P(D)y = f(t) \) for a particular solution \( y_p(t) \) where \( f \) is the following:
   (i) \( f(t) = 2e^{-t}\sin(2t) \)  (ii) \( f(t) = 2e^{-t}\cos(2t) \)  (iii) \( f(t) = t + 3 \).

(d) For \( b = 2 \), find the general solution of the DE \( P(D)y = 0 \)

Problem 7
Consider the DE \( y'' + 2y' + ky = 0 \).

(a) For which values of \( k \) will there be solutions \( y(t) \) with infinitely many zeros?

(b) For such solutions, express in terms of \( k \) the \( t \)-distance between successive zeros.

(c) For which values of \( k \) will \( \lim_{t \to \infty} y(t) = 0 \) for all solutions \( y(t) \)? (Indicate reason.)

(d) For which values of \( k \) will \( \lim_{t \to \infty} y(t) = 0 \) for at least one non-trivial solution \( y(t) \)? Why?

(e) Suppose in the DE the right-hand side is replaced by \( \sin(\omega t) \), and \( k = 10 \), let \( y_p(t) \) be a sinusoidal solution. Given that this system is lightly damped, and ignoring the damping: tell, without solving the DE, what integer frequency \( \omega_{int} \) will make the amplitude of \( y_p(t) \) relatively large?

Then, taking into account the fact that the system is in fact damped: to make the amplitude of the response larger, should you increase or decrease \( \omega \) from the natural frequency?

Problem 8
Consider the DE \( x'' + bx' + 5x = \cos(\omega t) \).

(a) For what \( b \) is it possible for this system to undergo pure resonance?

(b) For all the \( b \) values in your answer to part (a) give the corresponding resonant frequency.

(c) For each of the \( b \) values in the answer to part (a) solve the DE with \( \omega \) equal to the resonant frequency.

(d) Graph the amplitude response of each of the systems that can undergo resonance.

(e) Graph one nicely chosen solution from part (c).

(f) (Unrelated to the DE above.) Assume we have a (possibly frictionless) physical system modeled by a second order constant coefficient linear DE. What criteria must the roots satisfy if this system can undergo pure resonance?

Problem 9
Assume $L$ is a linear differential operator and $y_1$ is a solution to the DE $Ly = 0$. Prove that if $y_p$ is a solution to the DE $Ly = f$, then so are all the functions $y_p + cy_1$, where $c$ is any constant.

**Problem 10**

Suppose that $P(r)$ is a polynomial and consider the DE

$$P(D)x = \cos t + \cos 2t + \cos 3t.$$ 

The graph of $\frac{1}{|P(i\omega)|}$ is shown. On the axes provided give a rough sketch of the periodic solution $x_p$ to the DE. Give a brief explanation of your reasoning.

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*End of practice exam*