ES.1803 Problem Set 4 extra credit, Fall 2018

EC 1 Extra credit: Ballet of the Bugs (4: 1,1,1,1)

Four artistic bugs (idealized as points, of course) are initially located at the four vertices of a square. When the music starts, each bug starts crawling with its little snout always pointed at the bug on its right (and all at the same constant speed). The problem is to find the track of the bugs, by setting up a DE and solving it. We’ll be specific and take the initial four vertices to be (-1,-1), (1,-1), (1,1), (-1,1).

First, make a diagram and get as far as you can - in, say 15 minutes or so - with the set up needed. (OK - we’ll have to take your word for it; but you will get more out of the problem and also quite possibly save some time in the subsequent work if you do this.) Try to predict what the tracks of the bugs will look like in general terms. Then proceed as indicated below.

(a) Show that if \((x,y)\) is the current position of bug 1 on \(B_1\) then \((-y,x)\) is the position of bug 2 on \(B_2\).

(The famous 2D-vector fact that \((a,b)\perp = \pm(-b,a)\) may be of some interest here.)

(b) Use the result of part (a) and the direction of the bugs’ snouts to derive the following DE for \(B_1\): \[
\frac{dy}{dx} = \frac{x-y}{-y-x}\quad \text{with IC } y(-1) = -1.
\]

(c) Solve the DE in part (b).

(d) Show that in polar coordinates the solution you found in part (c) has the simple form \(r = Ke^{-\theta}\).

EC-2 Extra Credit: Pursuit (2)

The setup: In a square yard the mail carrier is at one corner. At one adjacent corner is a dog and at the other adjacent corner is the gate. At time \(t = 0\) the dog, who as been straining at his tether, breaks free and starts chasing the carrier. Simultaneously the frightened and now fleeing mail carrier starts running straight towards the gate.

Assume the dog’s chase strategy is to always point at the carrier, the dog runs at constant speed \(v_d\) and the carrier runs at constant speed \(v_m\). Finally assume the side length of the square yard is \(a\).

Decide the criteria on the parameters that will get the carrier safely out the gate.

For amusement: see why this problem is similar to the flight trajectories example in section 1.6 of the textbook.

EC-3 Extra Credit: The End of the Universe (4: 1,1,1,1)

In this problem there is a function \(R = R(t)\) which satisfies the IVP

\[
2R\frac{d^2R}{dt^2} + (R')^2 + c^2 = 0; \quad R(0) = R_0, \quad R'(0) = v_0.
\]

This equation has an interesting history in cosmology.

The ‘3D sphere’ of radius \(R\) is the three-dimensional space given, for example, by the equation \(x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\). This is the 3D analog of the usual 2D sphere \(x^2 + y^2 + z^2 = R^2\). In 1922, the Russian mathematical physicist A.A. Friedmann found a solution to Einstein’s (then new) general relativistic field equations for a very simplified cosmological
model which gives interesting results in the case of a 3D spherical universe. He assumed that, taken as whole, the matter in the universe is approximately homogeneous; this allowed him to derive the above ODE for the radius of the universe. Here \( R = R(t) \) is the radius of the spherical universe at time \( t \) and \( c \) is the speed of light. For IC’s, let \( t = 0 \) be now, so \( R_0 \) is the current radius and \( v_0 \) is the current rate of expansion. Clearly, \( R_0 > 0 \) and observation indicates that \( v_0 > 0 \) as well (Hubble red-shift and all that). We’ll solve this DE and see what it would imply for our very long-range future.

(a) We can reduce the order of this DE by multiplying the equation by \( R' \) and observing the left hand side can be written as \( \frac{d}{dt}(\text{terms involving only } R \text{ and } R') \). Antidifferentiate this to get a first order DE. Then use the IC to specify the constant of integration.

(b) Assume \( v_0 < c \). (The notations \( \alpha = \frac{v_0}{c} \) and \( K = R_0(1 + \alpha^2) \) will be helpful.) Now for the hard part: solve this DE with the given IC.

Hint: find a substitution \( R = f(u) \), that makes the DE separable. After solving you will have \( R \) and \( t \) defined parametrically in terms of \( u \).

(c) We’ll choose units so that \( c = 1 \). Take \( v_0 = 0.25c, \ R_0 = 8 \) and plot the parametric curve for \( R \text{ vs. } t \) over a reasonable range of \( u \).

(d) Using the results obtained above, show that this model of the universe predicts a periodic universe: a 'big bang' expansion, starting from \( R = 0 \) out to a maximum value for \( R \), followed by a 'big crunch' phase, during which the universe contracts from its maximum size back down to \( R = 0 \); followed by another big bang phase, ad ininitum. Also give the formula for the maximum size of the universe in terms of \( R_0 \) and \( v_0 \).